SOME BOUNDS FOR RAMIFICATION OF p^n-TORSION SEMI-STABLE REPRESENTATIONS

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Abstract. Let p be an odd prime, K a finite extension of $\mathbb{Q}_p$, $G_K = \mathrm{Gal}(\bar{K}/K)$ its absolute Galois group and $e = e(K/\mathbb{Q}_p)$ its absolute ramification index. Suppose that T is a p^n-torsion representation of G_K that is isomorphic to a quotient of G_K-stable $\mathbb{Z}_p$-lattices in a semi-stable representation with Hodge-Tate weights $\{0, \dots, r\}$. We prove that there exists a constant μ depending only on n, e and r such that the upper numbering ramification group $G_K^{(\mu)}$ acts on T trivially.

Contents

1. Introduction

Let $p > 2$ be a prime number and k a perfect field of characteristic p. We denote by $W = W(k)$ the ring of Witt vectors with coefficients in k. Fix a totally ramified extension K of $W[1/p]$ of degree e and an algebraic closure $\bar{K}$ of K. Set $G = \mathrm{Gal}(\bar{K}/K)$. Denote by $G^{(\mu)}$ ($\mu \in \mathbb{R}$) the upper ramification filtration of G, as defined in §1.1 of [10]. Note that conventions of *loc. cit.* differ from those by some shift from the definitions of [24], Chap. IV. Finally, let v_K be the discrete valuation on K normalized by $v_K(K^\star) = \mathbb{Z}$. It extends uniquely to a (nondiscrete) valuation on $\bar{K}$, which we denote again by v_K.

Consider r a positive integer and V a semistable representation of G with Hodge-Tate weights in $\{0, 1, \dots, r\}$. Let T be the quotient of two G-stable $\mathbb{Z}_p$-lattices in V. It is a representation of G,

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which is annihilated by p^n for some integer n. Denote by $\rho : G \to \mathrm{Aut}_{\mathbb{Z}_p}(T)$ the associated group homomorphism and by L the finite extension of K defined by $\ker \rho$. The main result of this paper is the following theorem.

Theorem 1.1. *Write $\frac{nr}{p-1} = p^\alpha \beta$ with $\alpha \in \mathbb{N}$ and $\frac{1}{p} < \beta \leq 1$. Then:*

(1) *if $\mu > 1 + e(n+\alpha) + \max(e\beta - \frac{1}{p^{n+\alpha}}, \frac{e}{p-1})$, then $G^{(\mu)}$ acts trivially on T;*
(2) *$v_K(\mathcal{D}_{L/K}) < 1 + e(n+\alpha+\beta) - \frac{1}{p^{n+\alpha}}$,*

where $\mathcal{D}_{L/K}$ is the different of L/K.

Before this work, some partial results were already known in this direction. First, in [10] and [12], Fontaine uses Fontaine-Laffaille's theory (developed in [9]) to get some bounds when $e = 1$, $n = 1$, $r < p - 1$ and V is crystalline. In [1], Abrashkin follows Fontaine's general ideas to extend the result to arbitrary n (other restrictions remain the same). Later, with the extension by Breuil of Fontaine-Laffaille's theory to semistable case (see [3]), it has been possible to achieve some cases where V is not crystalline. Precisely in [4] (see Proposition 9.2.2.2 of [2] for the statement), Breuil obtains bounds for semistable representations that satisfy Griffith transversality when $n = 1$ and $er < p - 1$. Recently in [15] and [16], Hattori proves a bound for all semistable representations with $r < p-1$ (e and n are arbitrary here). All these bounds (for μ and $v_K(\mathcal{D}_{L/K})$) have the same form

$$e\left(n + \frac{r}{p-1}\right) + (\text{some other terms}), \tag{1.0.1}$$

where the contribution of other terms that appear is always between 0 and 1. Since r is always assumed to be less than $p - 1$, one can see that these bounds are better than ours. However, the most important feature of Theorem 1.1 is to be applicable for any r! Furthermore, one remark that the bounds of Theorem 1.1 have a logarithmic dependence in r, which may be quite surprising after (1.0.1) where the dependence seems to be linear (of course, it does not mean anything since these bounds are conditional to the assumption $r < p - 1$). Actually, it is very plausible that, using analogous methods, one can improve Theorem 1.1 in order to fit with (1.0.1). Precisely, we conjecture the following.

Conjecture 1.2. *Writing $\frac{r}{p-1} = p^{\alpha'} \beta'$ with $\alpha' \in \mathbb{N}$ and $\frac{1}{p} < \beta' \leq 1$, we have:*

(1) *if $\mu > 1 + e(n+\alpha') + \max(e\beta' - \frac{1}{p^{n+\alpha'}}, \frac{e}{p-1})$ then $G^{(\mu)}$ acts trivially on T;*
(2) *$v_K(\mathcal{D}_{L/K}) < 1 + e(n+\alpha'+\beta') - \frac{1}{p^{n+\alpha'}}$.*

We finally wonder if better bounds exist when V is crystalline. This is the case with $e = 1$ and $r < p - 1$ by results of Fontaine and Abrashkin, but it is not clear to us how to extend this to a more general setting.

We would like to emphasize that Theorem 1.1 can be applied to many representations coming from geometry. For instance, by a famous theorem (see for instance [25]), we know that for any proper smooth X over K with semistable reduction over $\mathcal{O}_K$ and any non-negative integer r, the étale p-adic cohomology group $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$ is a semistable representation with Hodge-Tate weights in $\{0, \dots, r\}$. Hence, the bounds of Theorem 1.1 are valid for instance for the p^n-torsion representation L/p^nL where $L = H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}_p)/\text{torsion}$. One may wonder if the same bounds hold for $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}_p)/p^n$ or even for $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}/p^n\mathbb{Z})$. When $er < p-1$, it follows from Hattori's result [16] and a comparison theorem between étale and log-crystalline cohomology due to one of us [7]. In general, it seems to be unknown.

Another interesting example comes from the theory of modular forms since we know that the p-adic Galois representation associated to a modular form of level prime to p is always crystalline (and hence semistable).

Let us now explain the general plan of our proof (and at the same time of the article). For this we first introduce further notations: we fix a uniformizer $\pi \in \mathcal{O}_K$ and a compatible system $(\pi_s)_{s \geq 0}$ of p^s-th root of π. For all nonnegative integer s, put $K_s = K(\pi_s)$ and $G_s = \mathrm{Gal}(\bar{K}/K_s)$. Define also $K_\infty = \bigcup_{s=1}^{\infty} K_s$ and $G_\infty = \mathrm{Gal}(\bar{K}/K_\infty)$.

The general strategy follows closely the method introduced first by Fontaine to deal with this kind of questions; the main tool is to use some p-adic Hodge theory to describe quotients of lattices in semistable representations in terms of "objects of linear algebra". Unfortunately, in the generality of Theorem 1.1 — that is, for arbitrary r —, we do not have a complete machinery for that. Instead of that, we use Breuil-Kisin theory (suggested by Breuil in [5] and [6] and developped by Kisin in [18] and [19]) which gives just a description of the action of the subgroup G_∞ by some "Kisin modules"[1].

However, unfortunately, it is not true that $G_\infty \subset G^{(\mu)}$ for some μ. Hence studying only the action of G_∞ is certainly not enough to derive Theorem 1.1! That is the reason why we need to extend Breuil-Kisin theory a bit in the torsion case: we will actually show that if $\mathfrak{M}$ is the Kisin module associated to a L (with notation of Theorem 1.1), then the data of $\mathfrak{M}$ is enough to recover the whole action of G_s on T for $s > s_{\min} := n - 1 + \log_p(nr)$. More precisely, we prove in §2 that any Kisin module of p^n-torsion determines a representation of G_s with $s > s_{\min}$ (and not only G_∞). Note that this first step does not use any assumption of semistability: our result is true for all representations (annihilated by p^n) coming from a Kisin module, no matter if they can be realized as a quotient of two lattices in a semistable representation. Then, in §3, we show that the G_s-representation attached to $\mathfrak{M}$ coincide with $T|_{G_s}$. At this level, let us mention an interesting corollary of the theory developed in these two sections:

Theorem 1.3 (Corollary 3.3.5). *Let V and V' be two semistable representations of G with Hodge-Tate weights in $\{0, \dots, r\}$. Let T (resp. T') a quotient of two G-stable lattices in V (resp. V') which is annihilated by p^n. Then any G_∞-equivariant morphism $f : T \to T'$ is G_s-equivariant for all integer $s > n - 1 + \log_p(nr)$.*

Then, using usual techniques developed by Fontaine in [10], we prove the following Theorem, from which Theorem 1.1 easily follows using some kind of transivity formulas.

Theorem 1.4. *Keeping previous notations, for any integer $s > n + \log_p(\frac{nr}{p-1})$ and for all real number $\mu > \frac{ernp^n}{p-1}$, $G_s^{(\mu)}$ acts trivially on T.*

Remark 1.5. The condition on s implies $\frac{ernp^n}{p-1} < ep^s$. Hence one may always choose $\mu = ep^s$.

Finally, in §5, we begin a discussion about the possibility of writing a given torsion representation of G_K as a quotient of two lattices in a $\mathbb{Q}_p$-representation satisfying some properties (like being crystalline, semistable, with prescribed Hodge-Tate weights).

Conventions. For any $\mathbb{Z}$-module M, we always use M_n to denote M/p^nM. If A is a ring, then $\mathrm{M}_d(A)$ will denote the ring of $d \times d$-matrices with coefficients in A. We reserve φ to represent various Frobenius structures (except that σ stands for the usual Frobenius on $W(k)$) and φ_M will denote the Frobenius on M. But we always drop the subscript if no confusion arises.

Finally, if A is a ring equipped with a valuation v_A we will often set:

$$\mathfrak{a}_A^{\geqslant v} = \{x \in A \,/\, v_A(x) \geq v\} \quad \text{and} \quad \mathfrak{a}_A^{> v} = \{x \in A \,/\, v_A(x) > v\}.$$

2. G_s-representation attached to a torsion Kisin module

In this section, we prove that the G_∞-representation $T_{\mathfrak{S}_n}(\mathfrak{M})$ attached to Kisin modules $\mathfrak{M}$ annihilated by p^n can be naturally extended to a G_s-representation for all $s > n - 1 + \log_p(nr)$ (and sometimes better).

2.1. Definitions and basic properties of Kisin modules.

Recall the following notations: k is a perfect field, $W = W(k)$, K is a totally ramified extension of $W[1/p]$ of degree e, π is a fixed uniformizer of K. Recall also that we have fixed a positive integer r. Define $E(u)$ to be the minimal polynomial of π over $W[1/p]$.

[1] As we have already said, these modules were first introduced by Breuil. However, we think that this terminology is not so bad since "Breuil modules" already have a different meaning and "Kisin modules" were actually really studied by Kisin.

The base ring for Kisin modules is $\mathfrak{S} = W[\![u]\!]$. It is endowed with a Frobenius map $\varphi : \mathfrak{S} \to \mathfrak{S}$ defined by:

$$\varphi\Big(\sum_{i\geq 0} a_i u^i\Big) = \sum_{i\geq 0} \sigma(a_i) u^{pi}$$

where σ stands for usual Frobenius on W. By definition, a *free Kisin module* (of height $\leq r$) is a $\mathfrak{S}$-module $\mathfrak{M}$ free of finite rank equipped with a φ-semilinear endomorphism $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ such that the following condition holds:

(2.1.1) the $\mathfrak{S}$-submodule of $\mathfrak{M}$ generated by $\varphi_{\mathfrak{M}}(\mathfrak{M})$ contains $E(u)^r \mathfrak{M}$.

We denote by $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ their category. Of course, a morphism of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ is just a $\mathfrak{S}$-linear map that commutes with Frobenius actions. In the sequel, if there is no risk of confusion, we will often write φ instead of $\varphi_{\mathfrak{M}}$.

There is also a notion of *torsion Kisin modules* of height $\leq r$. They are modules $\mathfrak{M}$ over $\mathfrak{S}$ equipped with a φ-semilinear map $\varphi : \mathfrak{M} \to \mathfrak{M}$ such that:

- $\mathfrak{M}$ is finitely generated and annihilated by a power of p;
- $\mathfrak{M}$ has no u-torsion;
- condition (2.1.1) holds.

Let us call $\mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ (resp. $\mathrm{Mod}_{\mathfrak{S}_n}^{\varphi,r}$, resp. $\mathrm{Free}_{\mathfrak{S}_n}^{\varphi,r}$) the category of all torsion Kisin modules (resp. of torsion Kisin modules annihilated by p^n, resp. torsion Kisin modules annihilated by p^n and free over $\mathfrak{S}_n = \mathfrak{S}/p^n\mathfrak{S}$). Obviously $\mathrm{Free}_{\mathfrak{S}_n}^{\varphi,r} \subset \mathrm{Mod}_{\mathfrak{S}_n}^{\varphi,r}$ and $\bigcup_{n\geq 1} \mathrm{Mod}_{\mathfrak{S}_n}^{\varphi,r} = \mathrm{Mod}_{\mathfrak{S}_\infty}^{\varphi,r}$ (the union is increasing). It is proved in Proposition 2.3.2 of [20] that torsion Kisin modules are exactly quotients of two free Kisin modules of same rank. In particular every object in $\mathrm{Mod}_{\mathfrak{S}_n}^{\varphi,r}$ is a quotient of an object in $\mathrm{Free}_{\mathfrak{S}_n}^{\varphi,r}$. We finally note that *dévissages* with torsion Kisin modules are in general quite easy to achieve since if $\mathfrak{M}$ is in $\mathrm{Mod}_{\mathfrak{S}_n}^{\varphi,r}$ then $\mathfrak{M}_{(p)} = \ker p|_{\mathfrak{M}}$ and $\mathfrak{M}/\mathfrak{M}_{(p)}$ are respectively in $\mathrm{Mod}_{\mathfrak{S}_1}^{\varphi,r}$ and $\mathrm{Mod}_{\mathfrak{S}_{n-1}}^{\varphi,r}$ and we obviously have an exact sequence $0 \to \mathfrak{M}_{(p)} \to \mathfrak{M} \to \mathfrak{M}/\mathfrak{M}_{(p)} \to 0$ (see Proposition 2.3.2 in [20]).

2.2. **Functors to Galois representations.** We first need to define some period rings. Let $R = \varprojlim_s \mathcal{O}_{\bar{K}}/p$ where transition maps are Frobenius. By definition an element $x \in R$ is a sequence $(x^{(0)}, x^{(1)}, \ldots)$ such that $(x^{(s+1)})^p = x^{(s)}$. Fontaine proves in [13] that R is equipped with a valuation defined by $v_R(x) = \lim_{s\to\infty} p^s v_K(x^{(s)})$ if $x \neq 0$. (In this case, $x^{(s)}$ does not vanish for s large enough and its valuation is then well defined; starting from this rank, the sequence $p^s v_K(x^{(s)})$ is constant.) Note that k embeds naturally in R *via* $\lambda \mapsto (\lambda^{(0)}, \lambda^{(1)}, \ldots)$ where $\lambda^{(s)}$ is the unique p^s-th root of λ in k (recall that k is assumed to be perfect). This embedding turns R into a k-algebra. Now, consider $W(R)$ (resp. $W_n(R)$) the ring of Witt vectors (resp. truncated Witt vectors) with coefficients in R. It is a W-algebra (resp. a $W_n(k)$-algebra). Moreover, since Frobenius is bijective on R, $W_n(R) = W(R)/p^n W(R)$. Recall that we have fixed (π_s) a compatible sequence of p^s-roots of π. It defines an element $\underline{\pi} \in R$ whose Teichmüller representative is denoted by $[\underline{\pi}]$. We can then define an embedding $\mathfrak{S} \hookrightarrow W(R)$, $u \mapsto [\underline{\pi}]$. For any positive integer n, reducing modulo p^n, we get a map $\mathfrak{S}_n \hookrightarrow W_n(R)$ which remains injective. In the sequel, we shall still often denote by u its image in $W(R)$ and $W_n(R)$. Let $\mathcal{O}_{\mathcal{E}}$ be the closure in $W(\mathrm{Frac}\, R)$ of $\mathfrak{S}[1/u]$ (for the p-adic topology). Define $\mathcal{E} = \mathrm{Frac}\mathcal{O}_{\mathcal{E}}$ and $\widehat{\mathcal{E}}^{\mathrm{ur}}$ the p-adic completion of the maximal (algebraic) unramified extension of $\mathcal{E}$ in $W(\mathrm{Frac}\, R)[1/p]$. Denote $\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}$ its ring of integers and put $\mathfrak{S}^{\mathrm{ur}} = W(R) \cap \mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}$. Clearly $\mathfrak{S}^{\mathrm{ur}}$ is a subring of $W(R)$ and one can check (see Proposition 2.2.1 of [20]) that it induces an embedding $\mathfrak{S}_n^{\mathrm{ur}} = \mathfrak{S}^{\mathrm{ur}}/p^n\mathfrak{S}^{\mathrm{ur}} \hookrightarrow W_n(R)$. Remark finally that all previous rings are endowed with a Frobenius action.

Recall that G (resp. G_s) is the absolute Galois group of K (resp. $K_s = K(\pi_s)$) and that G_∞ is the intersection of all G_s. Denote by $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{free}}(G_\infty)$ (resp. $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{tor}}(G_\infty)$) the category of free (resp. torsion) $\mathbb{Z}_p$-representations of G_∞. We define functors $T_{\mathfrak{S}} : \mathrm{Mod}_{\mathfrak{S}}^{\varphi,r} \to \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{free}}(G_\infty)$ and $T_{\mathfrak{S}_n} : \mathrm{Mod}_{\mathfrak{S}_n}^{\varphi,r} \to \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{tor}}(G_\infty)$ by:

$$T_{\mathfrak{S}}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}) \quad \text{and} \quad T_{\mathfrak{S}_n}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}_n^{\mathrm{ur}})$$

where $\mathrm{Hom}_{\mathfrak{S},\varphi}$ means that we take all $\mathfrak{S}$-linear morphism that commutes with Frobenius. Note that $T_{\mathfrak{S}}(\mathfrak{M})$ and $T_{\mathfrak{S}_n}(\mathfrak{M})$ are *not* representations of G because this group does not act trivially on $\mathfrak{S} \subset W(R)$. If $n' \geq n$ then any object $\mathfrak{M}$ of $\mathrm{Mod}^{\varphi,r}_{\mathfrak{S}_n}$ is obviously also in $\mathrm{Mod}^{\varphi,r}_{\mathfrak{S}_{n'}}$ and we have a canonical identification $T_{\mathfrak{S}_n}(\mathfrak{M}) \simeq T_{\mathfrak{S}_{n'}}(\mathfrak{M})$. This fact allows us to glue all functors $T_{\mathfrak{S}_n}$ and define $T_{\mathfrak{S}_\infty} : \mathrm{Mod}^{\varphi,r}_{\mathfrak{S}_\infty} \to \mathrm{Rep}^{\mathrm{tor}}_{\mathbb{Z}_p}(G_\infty)$. An important result is the exactness of $T_{\mathfrak{S}_\infty}$ (see Corollary 2.3.4 of [20]).

Lemma 2.2.1 (Fontaine). *Let n be an integer and $\mathfrak{M}$ be an object of $\mathrm{Mod}^{\varphi,r}_{\mathfrak{S}_n}$. The embedding $\mathfrak{S}^{\mathrm{ur}}_n \hookrightarrow W_n(R)$ induces an isomorphism $T_{\mathfrak{S}_n}(\mathfrak{M}) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, W_n(R))$.*

Proof. See Proposition B.1.8.3 of [11]. □

2.3. **The modules $J_{n,c}(\mathfrak{M})$.** Let n be an integer and $\mathfrak{M}$ an object of $\mathrm{Mod}^{\varphi,r}_{\mathfrak{S}_n}$. For all nonnegative real number c, we define $\mathfrak{a}^{>c}_R = \{x \in R \,/\, v_R(x) > c\}$ and $[\mathfrak{a}^{>c}_R]$ the ideal of $W_n(R)$ generated by all $[x]$ with $x \in \mathfrak{a}^{>c}_R$ and, by the same way, $\mathfrak{a}^{\geqslant c}_R$ and $[\mathfrak{a}^{\geqslant c}_R]$. We have very explicit descriptions of these ideals:

Lemma 2.3.1. *Let $c \in \mathbb{R}^+$. Then:*

1. *for all $x_0, \ldots, x_{n-1} \in R$, $(x_0, \ldots, x_{n-1})$ is in $[\mathfrak{a}^{>c}_R]$ (resp $[\mathfrak{a}^{\geqslant c}_R]$) if and only if $v_R(x_i) > p^i c$ (resp. $v_R(x_i) \geq p^i c$) for all i ;*
2. *if $\gamma \in R$ has valuation c, then $[\mathfrak{a}^{\geqslant c}_R]$ is the principal ideal generated by $[\gamma]$.*

Proof. Easy with the formula $[z](x_0, \ldots, x_{n-1}) = (zx_0, z^p x_1, \ldots, z^{p^{n-1}} x_{n-1})$. □

Since $[\mathfrak{a}^{>c}_R]$ is stable under φ and G-action, the quotient $W_n(R)/[\mathfrak{a}^{>c}_R]$ inherits a Frobenius action and it makes sense to define:

$$J_{n,c}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, W_n(R)/[\mathfrak{a}^{>c}_R]). \tag{2.3.1}$$

It is endowed with an action of G_∞. Let's also denote $J_{n,\infty} = \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, W_n(R)) \simeq T_{\mathfrak{S}_n}(\mathfrak{M})$ (Lemma 2.2.1). Obviously, if $c \leq c' \leq \infty$, the reduction modulo $[\mathfrak{a}^{>c}_R]$ defines a natural G_∞-equivariant morphism $\rho_{c',c} : J_{n,c'}(\mathfrak{M}) \to J_{n,c}(\mathfrak{M})$. If $c \leq c' \leq c'' \leq \infty$, we have $\rho_{c'',c} = \rho_{c',c} \circ \rho_{c'',c'}$.

Lemma 2.3.2. *u is nilpotent in $W_n[u]/E(u)^r$.*

Proof. Since $E(u)$ is an Eisenstein polynomial, the congruence $E(u) \equiv u^e \pmod{p}$ holds in $W[u]$. Hence $E(u)^r \equiv u^{er} \pmod{p}$, which means that u^{er} is divisible by p in $W[u]/E(u)^r$. It follows that p^n divides u^{ern} in $W[u]/E(u)^r$, *i.e.* u^{ern} vanishes in $W_n[u]/E(u)^r$. □

Fix a positive integer N such that $u^N = 0$ in $W_n[u]/E(u)^r$. By previous proof one can take $N = ern$, but in many situations this exponent can be improved. In the following subsection, we will examine several examples. *From now on, we put $b = \frac{N}{p-1}$ and $a = b + N = \frac{pN}{p-1}$.*

Proposition 2.3.3. *The morphism $\rho_{\infty,b} : T_{\mathfrak{S}_n}(\mathfrak{M}) \to J_{n,b}(\mathfrak{M})$ is injective and its image is $\rho_{a,b}(J_{n,a}(\mathfrak{M}))$.*

Proof. We first prove injectivity. Let $f : \mathfrak{M} \to [\mathfrak{a}^{>b}_R]$ be a φ-morphism. We want to show that $f = 0$. First, remark that since $\mathfrak{M}$ is finitely generated, f takes its values in $[\mathfrak{a}^{\geqslant b'}_R]$ for some $b' > b$. Let $x \in \mathfrak{M}$. By definition of N, $u^N x$ belongs to $E(u)^r \mathfrak{M}$. By condition (2.1.1) we can write $u^N x = \lambda_1 \varphi(x_1) + \cdots + \lambda_k \varphi(x_k)$. Applying f, we get:

$$u^N f(x) = \lambda_1 \varphi(f(x_1)) + \cdots + \lambda_k \varphi(f(x_k)) \in [\mathfrak{a}^{\geqslant pb'}_R]$$

and then $f(x) \in [\mathfrak{a}^{\geqslant pb' - N}_R]$ (since $u = [\underline{\pi}]$). Applying the argument repeatedly, we see that $f(\mathfrak{M}) \subset \bigcap_{i \geq 0} [\mathfrak{a}^{\geqslant b_i}_R] W_n(R)$ where (b_i) is the sequence defined by $b_0 = b'$ and $b_{i+1} = pb_i - N$. Now $b' > \frac{N}{p-1}$ implies $\lim_{i \to \infty} b_i = \infty$. Injectivity follows.

Let's prove the second part of the proposition. Since $\rho_{\infty,b}$ factors through $\rho_{a,b}$ we certainly have $\rho_{\infty,b}(J_{n,\infty}) \subset \rho_{a,b}(J_{n,a})$. Conversely, we want to prove that if $f : \mathfrak{M} \to W_n(R)/[\mathfrak{a}^{>a}_R]$ is a φ-morphism, then there exists a φ-morphism (necessarily unique) $g : \mathfrak{M} \to W_n(R)$ such that $g \equiv f \pmod{[\mathfrak{a}^{>b}_R]}$. Assume first $\mathfrak{M} \in \mathrm{Free}^{\varphi,r}_{\mathfrak{S}_n}$ and pick a basis $(e_1, \ldots, e_d)$ of $\mathfrak{M}$ over $\mathfrak{S}_n$. Let A be a matrix with coefficients in $\mathfrak{S}_n$ such that:

$$(\varphi(e_1), \ldots, \varphi(e_d)) = (e_1, \ldots, e_d)A$$

and let X be a line vector with coefficients in $W_n(R)$ that lifts $(f(e_1), \dots, f(e_d))$.

The commutation of f and φ implies $XA \equiv \varphi(X) \pmod{[\mathfrak{a}_R^{\geqslant a}]}$. Actually, the congruence holds in $[\mathfrak{a}_R^{\geqslant a'}]$ for some $a' > a$. For the rest of the proof, fix some element $\alpha \in R$ with valuation a'. By Lemma 2.3.1.(2), $[\mathfrak{a}_R^{\geqslant a'}]$ is the principal ideal generated by $[\alpha]$. Therefore, one have $XA - \varphi(X) = -[\alpha]Q$ with coefficients of Q in $W_n(R)$. We want to prove that there exists a matrix Y with coefficients in $[\mathfrak{a}_R^{\geqslant b}]$ such that $(X+Y)A = \varphi(X+Y)$. Let us search Y of the shape $[\beta]Z$ with $\beta = \frac{\alpha}{u^N}$ (which belongs to R because of valuations) and coefficients of Z in $W_n(R)$. Our condition then becomes:

$$[\beta]ZA = [\beta^p]\varphi(Z) + [\alpha]Q \tag{2.3.2}$$

Using condition (2.1.1) and $u^N \in E(u)^r W_n[u]$, we find a matrix B (with coefficients in $\mathfrak{S}_n$) such that $BA = u^N$. Multiplying (2.3.2) by B on the left and simplifying by $[\alpha]$, we get the new equation:

$$Z = [\gamma]\varphi(Z)B + QB \tag{2.3.3}$$

with $\gamma = \alpha^{p-1}/u^{pN}$. Remark that $v_R(\gamma) = a'(p-1) - N > 0$; hence $\gamma \in R$. Now define a sequence (Z_i) by $Z_0 = 0$ and $Z_{i+1} = [\gamma]\varphi(Z_i)B + QB$. We have $Z_{i+1} - Z_i = [\gamma]\varphi(Z_i - Z_{i-1})B$. Since $v_R(\gamma) > 0$, $Z_{i+1} - Z_i$ goes to 0 for the u-adic topology (which is separate and complete on $W_n(R)$) when i goes to infinity. Hence (Z_i) converges to a limit Z which is solution of (2.3.3).

Finally, if $\mathfrak{M}$ is just an object of $\mathrm{Mod}_{\mathfrak{S}_n}^{\varphi,r}$ consider $\mathfrak{M}' \in \mathrm{Free}_{\mathfrak{S}_n}^{\varphi,r}$ and a surjective map $f : \mathfrak{M}' \to \mathfrak{M}$. Then $\ker f$ is in $\mathrm{Mod}_{\mathfrak{S}_n}^{\varphi,r}$ and sits in the following diagram:

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] \\
T_{\mathfrak{S}_n}(\mathfrak{M}) \arrow[r] \arrow[d] & J_{n,a}(\mathfrak{M}) \arrow[r] \arrow[d] & J_{n,b}(\mathfrak{M}) \arrow[d] \\
T_{\mathfrak{S}_n}(\mathfrak{M}') \arrow[r] \arrow[d] & J_{n,a}(\mathfrak{M}') \arrow[r] & J_{n,b}(\mathfrak{M}') \arrow[d] \\
T_{\mathfrak{S}_n}(\ker f) \arrow[rr, hook] & & J_{n,b}(\ker f)
\end{tikzcd}$$

All columns are exact (by left exactness of Hom) and the map on last line is injective (by first part of proposition). An easy diagram chase then ends the proof. $\square$

Remark 2.3.4. In general, $\rho_{a,b}$ is not surjective (nor injective) even for a and b big enough. Counter examples are very easy to produce: for instance, $\mathfrak{M} = \mathfrak{S}_1\mathfrak{e}$ equipped with $\varphi(\mathfrak{e}) = E(u)^r\mathfrak{e}$ is convenient.

2.4. **What is the best choice for N?** Here we are interested in finding integers N (as small as possible) such that $u^N = 0$ in $W_n[u]/E(u)^r$. As we have said before $N = ern$ is always convenient. If $n = 1$, it is obviously the best constant. However, it is no longer true for bigger n: the three following lemmas could give better exponents in many cases. We do not know how to find the best N in general.

In this paragraph, we will denote by $\lceil x \rceil$ the smallest integer not less than x.

Lemma 2.4.1. *We have $u^N = 0$ in $W_n[u]/E(u)^r$ for $N = ep^{n-1}\lceil \frac{r}{p^{n-1}} \rceil$.*

Proof. Just remark that $E(u)^{p^{n-1}} \equiv u^{ep^{n-1}} \pmod{p^n}$. $\square$

Lemma 2.4.2. *Assume $E(u) = u^e - p$. Then $u^N = 0$ in $W_n[u]/E(u)^r$ for $N = e(n+r-1)$.*

Remark 2.4.3. If $K/W[1/p]$ is tamely ramified, up to changing K by an unramified extension, we can always select an uniformizer whose minimal polynomial is $E(u) = u^e - p$.

Proof. Up to performing the variables change $v = u^e$, one may assume $e = 1$. We then have an isomorphism $f : K[u]/E(u)^r \to K^r$, $P \mapsto (P(p), P'(p), \dots, \frac{P^{(r-1)}(p)}{(r-1)!})$ whose inverse is given by

$f^{-1}(x_0,\dots,x_{r-1}) = x_0 + x_1(u-p) + \cdots + x_{r-1}(u-p)^{r-1}$. In particular $f(W[u]/E(u)^r) \supset W^r$. Moreover:

$$f(u^N) = \left(p^N, Np^{N-1}, \dots, \binom{N}{r-1} p^{N-r+1}\right) \in p^{N-r+1}W^r = p^n W^r.$$

Conclusion follows. □

Lemma 2.4.4. *There exists a constant c depending only on K such that $u^N = 0$ in $W_n[u]/E(u)^r$ for $N = en + c(r-1)$.*

Proof. The general plan of the proof is very similar to the previous one. We first consider the map $f : W[1/p][u]/E(u)^r \to K^r$, $P \mapsto (P(\pi), P'(\pi), \dots, \frac{P^{(r-1)}(\pi)}{(r-1)!})$. It is $W[1/p]$-linear and injective. Since both sides are $W[1/p]$-vector spaces of dimension er, f is an isomorphism. Denote by $\varpi \in W[1/p][u]/E(u)^r$ the preimage of $(\pi, 0, \dots, 0)$. The inverse of f is then given by the formula:

$$f^{-1}(x_0,\dots,x_{r-1}) = X_0(\varpi) + X_1(\varpi)(u-\varpi) + \cdots + X_{r-1}(\varpi)(u-\varpi)^{r-1}$$

where X_i are polynomials with coefficients in $W[1/p]$ such that $X_i(\pi) = x_i$. Second, we would like to bound below the "p-adic valuation" of $f^{-1}(x_0,\dots,x_{r-1})$ when all x_i's lies in $\mathcal{O}_K$. For that, we remark that $E(\varpi)$ is mapped to 0 by f; hence it vanishes. Solving this equation by successive approximations, we find that ϖ can be written $P_0(u) + P_1(u)E(u) + \cdots + P_{r-1}(u)E(u)^{r-1}$ with $P_0(u) = u$ and:

$$E'(u)P_i(u) \equiv \frac{E(P_0(u) + P_1(u)E(u) + \cdots + P_{i-1}(u)E(u)^{i-1})}{E(u)^i} \pmod{E(u)}$$

where P_i are uniquely determined modulo $E(u)^{r-i}$. Let $F(u) \in W[1/p][u]/E(u)$ be the inverse of $E'(u)$ and v an integer such that $p^v F(u) \in W[u]/E(u) \simeq \mathcal{O}_K$. (Note that $v = \lceil v_p(\mathcal{D}_{K/W[1/p]}) \rceil$ is convenient.) By induction we easily prove that $p^{iv}P_i(u) \in W[u]/E(u)^{r-i}$, and then that $Q(\varpi) \in W[u]/E(u)^r$ for all $Q \in p^{(r-1)v}W[u]$. Consequently $f(W[u]/E(u)^r) \supset p^{(r-1)v}\mathcal{O}_K^r$. Finally, defining $c = ev + 1$ and $N = en + c(r-1)$, we have:

$$f(u^N) = \left(\pi^N, N\pi^{N-1}, \dots, \binom{N}{r-1}\pi^{N-r+1}\right) \in \pi^{N-r+1}\cdot \mathcal{O}_K^r \subset p^{(r-1)v+n}\cdot \mathcal{O}_K^r$$

and we are done. □

2.5. **Some quotients of $W_n(R)$.** The aim of this last subsection is to study the structure of quotients $W_n(R)/[\mathfrak{a}_R^{>c}]$ that appear in the definition of $J_{n,c}$ (see formula (2.3.1)). It will allow us to derive interesting corollaries about the extension to a finite index subgroup of G of the natural action of G_∞ on $T_{\mathfrak{S}_n}(\mathfrak{M})$.

For a nonnegative integer s, let us denote by θ_s the ring morphism $R \to \mathcal{O}_{\bar{K}}/p$, $x = (x^{(0)}, x^{(1)}, \dots) \mapsto x^{(s)}$. We emphasize that it is not k-linear: it induces a morphism of k-algebras between R and $k \otimes_{k,\sigma^s} \mathcal{O}_{\bar{K}}/p$. For a nonnegative real number c, define:

$$\mathfrak{a}_{\bar{K}}^{>c} = \{x \in \bar{K} / v_K(x) > c\} \subset \mathcal{O}_{\bar{K}}.$$

Lemma 2.5.1. *Let c be a positive real number. For any integer $s > \log_p(\frac{c}{e})$, the map θ_s induces a Galois equivariant isomorphism of k-algebras*

$$R/\mathfrak{a}_R^{>c} \to k \otimes_{k,\sigma^s} \mathcal{O}_{\bar{K}}/\mathfrak{a}_{\bar{K}}^{>c/p^s}.$$

Proof. The map is clearly surjective. It remains to show that $x = (x^{(0)}, x^{(1)}, \dots)$ has valuation greater than c if and only if $v_K(x^{(s)}) > \frac{c}{p^s}$, which follows directly from $\frac{c}{p^s} < e$. □

Proposition 2.5.2. *Let c be a positive real number. For any $s > n - 1 + \log_p(\frac{c}{e})$, θ_s induces a G-equivariant isomorphism of $W_n(k)$-algebras:*

$$\frac{W_n(R)}{[\mathfrak{a}_R^{>c}]} \to W_n(k) \otimes_{W_n(k),\sigma^s} \frac{W_n(\mathcal{O}_{\bar{K}}/p)}{[\mathfrak{a}_{\bar{K}}^{>c/p^s}]}.$$

Proof. Since θ_s is surjective, the above map is also surjective. Let $x = (x_0, \dots, x_{n-1}) \in W_n(R)$ and assume that $x^{(s)} = (x_0^{(s)}, \dots, x_{n-1}^{(s)})$ lies in $[\mathfrak{a}_{\bar{K}}^{>c/p^s}]$. By an analogue of Lemma 2.3.1.(1), one obtains $v_K(x_i^{(s)}) > \frac{c}{p^{s-i}}$ for all i. Hence, $x_i^{(s)}$ is in $\mathfrak{a}_{\bar{K}}^{>cp^i/p^s}$. Since $\log_p(\frac{cp^i}{e}) = i + \log_p(\frac{c}{e}) \leq n-1+\log_p(\frac{c}{e}) < s$, we can apply Lemma 2.5.1 and deduce $x_i \in [\mathfrak{a}_R^{>cp^i}]$, *i.e.* $v_R(x_i) > cp^i$ for all i. By Lemma 2.3.1.(1), it follows that $x \in [\mathfrak{a}_R^{>c}]$. Thus, the map of the proposition is injective and we are done. □

Define increasing functions s_0 and s_1 by $s_0(c) = n-1+\log_p(\frac{c}{e})$ and $s_1(c) = n-1+\log_p(\frac{c(p-1)}{ep}) = s_0(c) + \log_p(1-\frac{1}{p})$. Recall that we have defined $a = \frac{pN}{p-1}$ (where N is an integer such that $u^N = 0$ in $W_n[u]/E(u)$) and set $s_{\min} = s_1(a) = n-1+\log_p(\frac{N}{e})$. If we choose $N = ern$, we just have $s_{\min} = n-1+\log_p(rn)$.

Proposition 2.5.3. *Let n be a positive integer and $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_n}^{\varphi,r}$. For any nonnegative integer $s > s_1(c)$, the natural action of G_s on $W_n(R)$ turns $J_{n,c}(\mathfrak{M})$ into a $\mathbb{Z}_p[G_s]$-module. Furthermore, we have the following compatibilities:*

- *the action of G_s is compatible with the usual action of G_∞ on $J_{n,c}(\mathfrak{M})$;*
- *if $s' \geq s \geq s_1(c)$, actions of $G_{s'}$ and G_s on $J_{n,c}$ are compatible;*
- *if $c' \geq c$ and $s \geq s_1(c')$, then $\rho_{c',c} : J_{n,c'}(\mathfrak{M}) \to J_{n,c}(\mathfrak{M})$ is G_s-equivariant.*

Proof. For the first statement, it is enough to show that G_s acts trivially on $u \in W_n(R)/[\mathfrak{a}_R^{>c}]$ for $s = 1+[s_1(c)]$ (where $[\cdot]$ denotes the integer part). Put $s' = 1+[s_0(c)]$. Since $0 \leq \log_p(\frac{p}{p-1}) \leq 1$, we have $s' = s$ or $s' = s+1$. By Proposition 2.5.2, $W_n(R)/[\mathfrak{a}_R^{>c}]$ is isomorphic to $W_n(\mathcal{O}_{\bar{K}}/p)/[\mathfrak{a}_{\bar{K}}^{>c/p^{s'}}]$. Hence we have to show that $g[\pi_{s'}] - [\pi_{s'}]$ belongs to $[\mathfrak{a}_R^{>c/p^{s'}}]$ for all $g \in G_s$. It is clear for $g \in G_{s'}$ (since the difference vanishes). It remains to consider the case where $s' = s+1$ and $g \notin G_{s'} = G_{s+1}$. Then $g\pi_{s+1} = (1+\eta)\pi_{s+1}$ where $(1+\eta)$ is a primitive p-th root of unity. Let us compute $(g\pi_{s+1}, 0, \dots, 0) - (\pi_{s+1}, 0, \dots, 0) = (x_0, \dots, x_{n-1})$ in $W_n(\mathcal{O}_{\bar{K}})$. By writing phantom components, we get the following system:

$$\begin{cases} x_0 = \eta\, \pi_{s+1} \\ x_0^p + px_1 = 0 \\ \quad \vdots \\ x_0^{p^{n-1}} + px_1^{p^{n-2}} + \cdots + p^{n-1}x_{n-1} = 0 \end{cases}$$

Using $v_K(\eta) = \frac{e}{p-1}$, we easily prove by induction on i that $v_K(x_i) = \frac{e}{p-1} + \frac{1}{p^{s+1-i}}$. Thus $v_K(x_i) > \frac{c}{p^{s+1-i}}$ for all i and $(x_0, \dots, x_{n-1}) \in [\mathfrak{a}_{\bar{K}}^{>c/p^{s'}}]$ as expected.

Second part of proposition (*i.e.* compatibilities) is obvious. □

Remark 2.5.4. If $c \geq \frac{p-1}{p-2}$, the bound $s_1(c)$ that appears in Proposition 2.5.2 can be replaced by $s_1(c-1)$. The proof is totally the same.

Theorem 2.5.5. *For any $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_n}^{\varphi,r}$ and any integer $s > s_{\min}$, $T_{\mathfrak{S}_n}(\mathfrak{M})$ is canonically endowed with an action of G_s (which prolongs the natural action of G_∞).*

Proof. Just combine Propositions 2.3.3 and Proposition 2.5.3. □

Remark 2.5.6. Using Remark 2.5.4, it appears that we may replace $s_{\min} = s_1(a)$ by $s_1(a-1)$ in previous Theorem. However, it won't be useful in the sequel since $s_{\min}$ is really needed in Theorem 3.3.4.

3. Torsion semistable Galois representations

In this section, we use the theory of $(\varphi, \hat{G})$-modules to define $\hat{J}_{n,a}(\hat{\mathfrak{M}})$ attached to p^n-torsion semistable representation T. After establishing isomorphism (of $\mathbb{Z}_p[G_\infty]$-modules) between $\hat{J}_{n,a}(\hat{\mathfrak{M}})$ and $J_{n,a}(\mathfrak{M})$, we will show that $J_{n,a}(\mathfrak{M}) \simeq T$ as G_s-modules for $s > s_{\min}$.

3.1. **Torsion $(\varphi, \hat{G})$-modules.** We refer readers to [14] for the definition and standard facts on semistable representations.

We first review some facts on $(\varphi, \hat{G})$-modules in [21] and extend them to p^n-torsion case. We denote by S the p-adic completion of the divided power envelope of $W(k)[u]$ with respect to the ideal generated by $E(u)$. There is a unique continuous map (Frobenius) $\varphi : S \to S$ which extends the Frobenius on $\mathfrak{S}$. Define a continuous $W(k)$-linear derivation $N : S \to S$ such that $N(u) = -u$.

Recall $R = \varprojlim_s \mathcal{O}_{\bar{K}}/p$. There is a unique surjective continuous map $\theta : W(R) \to \widehat{\mathcal{O}}_{\bar{K}}$ which lifts the projection $R \to \mathcal{O}_{\bar{K}}/p$ onto the first factor in the inverse limit. We denote by A_{cris} the p-adic completion of the divided power envelope of $W(R)$ with respect to $\mathrm{Ker}(\theta)$. Recall that $[\underline{\pi}] \in W(R)$ is the Teichmüller representative of $\underline{\pi} = (\pi_s)_{s \geq 0} \in R$ and we embed the $W(k)$-algebra $W(k)[u]$ into $W(R)$ via $u \mapsto [\underline{\pi}]$. Since $\theta(\underline{\pi}) = \pi$, this embedding extends to an embedding $\mathfrak{S} \hookrightarrow S \hookrightarrow A_{\mathrm{cris}}$, and $\theta|_S$ is the $W(k)$-linear map $s : S \to \mathcal{O}_K$ defined by sending u to π. The embedding is compatible with Frobenius endomorphisms. As usual, we write $B^+_{\mathrm{cris}} := A_{\mathrm{cris}}[1/p]$.

For any field extension $F/\mathbb{Q}_p$, set $F_{p^\infty} := \bigcup_{n=1}^{\infty} F(\zeta_{p^n})$ with ζ_{p^n} a primitive p^n-th root of unity. Note that $K_{\infty,p^\infty} := \bigcup_{n=1}^{\infty} K(\pi_n, \zeta_{p^n})$ is Galois over K. Let $G_{p^\infty} := \mathrm{Gal}(K_{\infty,p^\infty}/K_{p^\infty})$, $H_K := \mathrm{Gal}(K_{\infty,p^\infty}/K_\infty)$ and $\hat{G} := \mathrm{Gal}(K_{\infty,p^\infty}/K)$. By Lemma 5.1.2 in [22], we have $K_{p^\infty} \cap K_\infty = K$, $\hat{G} = G_{p^\infty} \rtimes H_K$ and $G_{p^\infty} \simeq \mathbb{Z}_p(1)$. For any $g \in G$, write $\underline{\epsilon}(g) = g(\underline{\pi})/\underline{\pi}$. Then $\underline{\epsilon}(g)$ is a cocycle from G to the group of units of R^*. In particular, fixing a topological generator τ of G_{p^∞}, the fact that $\hat{G} = G_{p^\infty} \rtimes H_K$ implies that $\underline{\epsilon}(\tau) = (\epsilon_s)_{s \geq 0} \in R^*$ with ϵ_s a *primitive* p^s-th root of unity. Therefore, $t := -\log([\underline{\epsilon}(\tau)]) \in A_{\mathrm{cris}}$ is well defined and for any $g \in G$, $g(t) = \chi(g)t$ where χ is the cyclotomic character. We reserve $\underline{\epsilon}$ for $\underline{\epsilon}(\tau)$.

For any integer $n \geq 0$, let $t^{\{n\}} = t^{r(n)} \gamma_{\tilde{q}(n)}(t^{p-1}/p)$ where $n = (p-1)\tilde{q}(n) + r(n)$ with $0 \leq r(n) < p-1$ and $\gamma_i(x) = \frac{x^i}{i!}$ is the standard divided power. Define subrings $\mathcal{R}_{K_0}$ and $\widehat{\mathcal{R}}$ of B^+_{cris} as in §2.2, [21]:

$$\mathcal{R}_{K_0} := \left\{ x = \sum_{i=0}^{\infty} f_i t^{\{i\}}, f_i \in S[1/p] \text{ and } f_i \to 0 \text{ as } i \to +\infty \right\}$$

and $\widehat{\mathcal{R}} := W(R) \cap \mathcal{R}_{K_0}$. Let $I_+R = \{x \in R \,/\, v_R(x) > 0\} = \mathfrak{a}_R^{>0}$ be the maximal ideal of R. We have exact sequences

$$0 \to W_n(I_+R) \to W_n(R) \xrightarrow{\nu_n} W_n(\bar{k}) \to 0 \text{ and } 0 \to W(I_+R) \to W(R) \xrightarrow{\nu} W(\bar{k}) \to 0$$

where ν_n are ν are induced by the composite $R \to \mathcal{O}_{\bar{K}}/p \to \bar{k}$, the first map being the projection onto the first factor in the inverse limit. One can naturally extend ν to $\nu : B^+_{\mathrm{cris}} \to W(\bar{k})[\frac{1}{p}]$ (see the proof of Lemma 2.2.1 in [21]). For any subring A of B^+_{cris} (resp. $W_n(R)$), we write $I_+A = \mathrm{Ker}(\nu) \cap A$ (resp. $I_+A = \mathrm{Ker}(\nu_n) \cap A$) and $I_+ := I_+\widehat{\mathcal{R}}$. Now recall M_n stands for M/p^nM.

Lemma 3.1.1. *We have the following commutative diagram :*

$$\begin{tikzcd} 0 \arrow[r] & W_n(I_+R) \arrow[r] & W_n(R) \arrow[r, "\nu_n"] & W_n(\bar{k}) \arrow[r] & 0 \\ 0 \arrow[r] & I_{+,n} \arrow[r] \arrow[u, hook] & \widehat{\mathcal{R}}_n \arrow[r] \arrow[u, hook] & W_n(k) \arrow[r] \arrow[u, hook] & 0 \end{tikzcd} \tag{3.1.1}$$

such that both rows are short exact and all vertical arrows are injective.

Proof. By Lemma 2.2.1 in [21], we have a commutative diagram of exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & I_+W(R) \arrow[r] & W(R) \arrow[r, "\nu"] & W(\bar{k}) \arrow[r] & 0 \\ 0 \arrow[r] & I_+ \arrow[r] \arrow[u, hook] & \widehat{\mathcal{R}} \arrow[r] \arrow[u, hook] & W(k) \arrow[r] \arrow[u, hook] & 0 \end{tikzcd}$$

Modulo p^n and noting that $I_+W(R) = I_+B^+_{\mathrm{cris}} \cap W(R) = W(I_+R)$, we get

$$\begin{tikzcd} 0 \arrow[r] & W_n(I_+R) \arrow[r] & W_n(R) \arrow[r, "\nu_n"] & W_n(\bar{k}) \arrow[r] & 0 \\ & I_{+,n} \arrow[r] \arrow[u] & \widehat{\mathcal{R}}_n \arrow[r] \arrow[u] & W_n(k) \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

Now it suffices to show that the bottom arrow is left exact and the last two vertical arrows are injective. The last one is obvious. To see the middle arrow is injective, it suffices to show that $(p^nW(R)) \cap \widehat{\mathcal{R}} = p^n\widehat{\mathcal{R}}$. Note that $\widehat{\mathcal{R}} = \mathcal{R}_{K_0} \cap W(R)$. Let $x \in W(R)$ such that $p^nx \in \widehat{\mathcal{R}} \subset \mathcal{R}_{K_0}$. Then $x \in W(R) \cap \mathcal{R}_{K_0} = \widehat{\mathcal{R}}$. So $p^nx \in p^n\widehat{\mathcal{R}}$ and $(p^nW(R)) \cap \widehat{\mathcal{R}} = p^n\widehat{\mathcal{R}}$. To see the bottom is left exact, it suffices to show that $I_+ \cap p^n\widehat{\mathcal{R}} = p^nI_+$. But $I_+ = I_+\mathcal{R}_{K_0} \cap \widehat{\mathcal{R}}$. Let $x \in \widehat{\mathcal{R}}$ such that $p^nx \in I_+ \cap p^n\widehat{\mathcal{R}}$. Then $x \in I_+\mathcal{R}_{K_0}$. Thus $x \in I_+\mathcal{R}_{K_0} \cap \widehat{\mathcal{R}} = I_+$ and $I_+ \cap p^n\widehat{\mathcal{R}} = p^nI_+$. $\square$

As in Lemma 2.2.1 in [21], we see that $\widehat{\mathcal{R}}$ (resp. $\widehat{\mathcal{R}}_n$) is φ-stable and G-stable subring of $W(R)$ (resp. $W_n(R)$), G-action on $\widehat{\mathcal{R}}$ factors through $\hat{G}$. Let $(\mathfrak{M}, \varphi)$ be a finite free or p^n-torsion Kisin module of height $\leq r$, set $\hat{\mathfrak{M}} := \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ and consider the following composite

$$\mathfrak{M} \simeq \mathfrak{S} \otimes_{\mathfrak{S}} \mathfrak{M} \to \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} = \hat{\mathfrak{M}} \tag{3.1.2}$$

where the first map is $\phi \otimes \mathrm{id}$. We claim that it is an injective (thus $\mathfrak{M}$ can be always regarded as a $\varphi(\mathfrak{S})$-submodule of $\hat{\mathfrak{M}}$). Indeed, by Lemma 3.1.1, we have $\varphi(\mathfrak{S}_n) \hookrightarrow \widehat{\mathcal{R}}_n \hookrightarrow W_n(R)$. Thus the claim is clear if $\mathfrak{M}$ is finite $\mathfrak{S}$-free or $\mathfrak{M}$ is finite $\mathfrak{S}_n$-free. For a general $\mathfrak{M}$ which is annihilated by p^n, by the discussion in the end of §2.1, $\mathfrak{M}$ can be written as a successive extension of finite free $\mathfrak{S}_1$-modules. Therefore one can reduce the proof of the claim to the following lemma.

Lemma 3.1.2. *The functor $\mathfrak{M} \mapsto \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ (resp. $\mathfrak{M} \mapsto W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$) is an exact functor from the category of Kisin modules to the category of $\widehat{\mathcal{R}}$-modules (resp. $W(R)$-modules).*

Proof. We only prove the exactness of the first functor, the proof for the second being totally identical. It suffices to prove that $\mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}, \widehat{\mathcal{R}}) = 0$ for any Kisin module $\mathfrak{M}$. Note that there exists finite free Kisin modules $\mathfrak{L}_1 \subset \mathfrak{L}_2$ such that $\mathfrak{M} = \mathfrak{L}_2/\mathfrak{L}_1$ (*cf* discussion in the end of §2.1). Since $\widehat{\mathcal{R}} \hookrightarrow W(R)$ is an integral domain and $\varphi : W(R) \to W(R)$ is injective, we see $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{L}_1 \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{L}_2$ is injective. Thus $\mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}, \widehat{\mathcal{R}}) = 0$. $\square$

Let $(\mathfrak{M}, \varphi)$ be a Kisin module of height $\leq r$ and $\hat{\mathfrak{M}} := \widehat{\mathcal{R}} \otimes_{\mathfrak{S},\varphi} \mathfrak{M}$. Frobenius φ on $\mathfrak{M}$ can be extended to $\hat{\mathfrak{M}}$ semilinearly by $\varphi_{\hat{\mathfrak{M}}}(a \otimes x) = \varphi_{\widehat{\mathcal{R}}}(a) \otimes \varphi_{\mathfrak{M}}(x)$.

Now we can make the following definition: a $(\varphi, \hat{G})$-*module of height* $\leq r$ is a triple $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ where

(1) $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a Kisin module of height $\leq r$;
(2) $\hat{G}$ is a $\widehat{\mathcal{R}}$-semilinear $\hat{G}$-action on $\hat{\mathfrak{M}} = \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$;
(3) $\hat{G}$ commutes with $\varphi_{\hat{\mathfrak{M}}}$ on $\hat{\mathfrak{M}}$, *i.e.* for any $g \in \hat{G}$, $g\varphi_{\hat{\mathfrak{M}}} = \varphi_{\hat{\mathfrak{M}}}g$;
(4) regard $\mathfrak{M}$ as a $\varphi(\mathfrak{S})$-submodule in $\hat{\mathfrak{M}}$, then $\mathfrak{M} \subset \hat{\mathfrak{M}}^{H_K}$;
(5) $\hat{G}$ acts on $W(k)$-module $M := \hat{\mathfrak{M}}/I_+\hat{\mathfrak{M}} \simeq \mathfrak{M}/u\mathfrak{M}$ trivially.

A morphism $f : (\mathfrak{M}, \varphi, \hat{G}) \to (\mathfrak{M}', \varphi', \hat{G}')$ is a morphism $\mathfrak{f} : (\mathfrak{M}, \varphi) \to (\mathfrak{M}', \varphi')$ of Kisin modules such that $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{f} : \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}'$ is $\hat{G}$-equivariant. If $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G})$ is a $(\varphi, \hat{G})$-module, we will often abuse notations by denoting $\hat{\mathfrak{M}}$ the underlying module $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. A $(\varphi, \hat{G})$-module $\hat{\mathfrak{M}} := (\mathfrak{M}, \varphi, \hat{G})$ is called *finite free* (resp. p^n-torsion) if $\mathfrak{M}$ is finite $\mathfrak{S}$-free (resp. $\mathfrak{M}$ is annihilated by p^n).

Let $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G})$ be a $(\varphi, \hat{G})$-module. We can associate $\mathbb{Z}_p[G]$-modules:

$$\hat{T}(\hat{\mathfrak{M}}) := \mathrm{Hom}_{\widehat{\mathcal{R}},\varphi}(\hat{\mathfrak{M}}, W(R)) \text{ if } \mathfrak{M} \text{ is finite } \mathfrak{S}\text{-free.}$$

and

$$\hat{T}_n(\hat{\mathfrak{M}}) := \mathrm{Hom}_{\widehat{\mathcal{R}},\varphi}(\hat{\mathfrak{M}}, W_n(R)) \text{ if } \mathfrak{M} \text{ is of } p^n\text{-torsion.}$$

Here G acts on $\hat{T}(\hat{\mathfrak{M}})$ (resp. $\hat{T}_n(\hat{\mathfrak{M}})$) via $g(f)(x) = g(f(g^{-1}(x)))$ for any $g \in G$ and $f \in \hat{T}(\hat{\mathfrak{M}})$ (resp. $\hat{T}_n(\hat{\mathfrak{M}})$). For any $f \in T_{\mathfrak{S}}(\mathfrak{M})$ (resp. $T_{\mathfrak{S}_n}(\mathfrak{M})$), set $\theta(f) \in \mathrm{Hom}_{\widehat{\mathcal{R}}}(\hat{\mathfrak{M}}, W(R))$ (resp. $\theta_n(f) \in \mathrm{Hom}_{\widehat{\mathcal{R}}}(\hat{\mathfrak{M}}, W_n(R))$) via:

$$\theta(f)(a \otimes x) \text{ (resp. } \theta_n(f)(a \otimes x)) = a\varphi(f(x)) \text{ for any } a \in \widehat{\mathcal{R}}, x \in \mathfrak{M}. \tag{3.1.3}$$

It is routine to check that $\theta : T_{\mathfrak{S}}(\mathfrak{M}) \to \hat{T}(\hat{\mathfrak{M}})$ (resp. $\theta_n : T_{\mathfrak{S}_n}(\mathfrak{M}) \to \hat{T}_n(\hat{\mathfrak{M}})$) is well-defined.

Denote by $\mathrm{Rep}_{\mathrm{tor}}(G)$ the category of G-representations on finite type $\mathbb{Z}_p$-modules which are annihilated by some p-power, and $\mathrm{Rep}^{\mathrm{ss},r}_{\mathrm{tor}}(G)$ the full subcategory of *torsion semistable representations* with Hodge-Tate weights in $\{0, \dots, r\}$ in the sense that there exist G-stable $\mathbb{Z}_p$-lattices $\Lambda_1 \subset \Lambda_2 \subset V$ such that V is semistable with Hodge-Tate weights in $\{0, \dots, r\}$ and $T \simeq \Lambda_2/\Lambda_1$ as $\mathbb{Z}_p[G]$-modules. The following is the main result of this subsection.

Theorem 3.1.3. (1) *Let $\hat{\mathfrak{M}} := (\mathfrak{M}, \varphi, \hat{G})$ be a $(\varphi, \hat{G})$-module. Then θ (resp. θ_n) induces a natural isomorphism of $\mathbb{Z}_p[G_\infty]$-modules $\theta : T_{\mathfrak{S}}(\mathfrak{M}) \xrightarrow{\sim} \hat{T}(\hat{\mathfrak{M}})$ (resp. $\theta_n : T_{\mathfrak{S}_n}(\mathfrak{M}) \xrightarrow{\sim} \hat{T}_n(\hat{\mathfrak{M}})$).*

(2) *$\hat{T}$ induces an anti-equivalence between the category of finite free $(\varphi, \hat{G})$-modules of height $\leq r$ and the category of G-stable $\mathbb{Z}_p$-lattices in semistable representations with Hodge-Tate weights in $\{0, \dots, r\}$.*

(3) *For any $T \in \mathrm{Rep}^{\mathrm{ss},r}_{\mathrm{tor}}(G)$, there exists a torsion $(\varphi, \hat{G})$-modules $\hat{\mathfrak{M}}$ such that $\hat{T}_n(\hat{\mathfrak{M}}) \simeq T$ as $\mathbb{Z}_p[G]$-modules for some n.*

Proof. (1) If $\mathfrak{M}$ is finite $\mathfrak{S}$-free then it has been proved in Theorem 2.3.1 in [21]. The proof of the p^n-torsion case is almost the same, except one need to check that $\mathfrak{M}$ is a $\varphi(\mathfrak{S})$-submodule of $\hat{\mathfrak{M}}$ via (3.1.2), which has been proved below (3.1.2).

(2) See Theorem 2.3.1 in [21].

(3) Let $\Lambda_1 \subset \Lambda_2$ be G-stable $\mathbb{Z}_p$-lattices inside a semistable representation with Hodge-Tate weights in $\{0, \dots, r\}$ such that $T \simeq \Lambda_2/\Lambda_1$ as $\mathbb{Z}_p[G]$-modules. By (2), there exists an injection of Kisin modules (resp. $(\varphi, \hat{G})$-modules) $i : \mathfrak{L}_2 \hookrightarrow \mathfrak{L}_1$ (resp. $\hat{i} : \hat{\mathfrak{L}}_2 \hookrightarrow \hat{\mathfrak{L}}_1$) that corresponds the inclusion $\Lambda_1 \subset \Lambda_2$. Write $\mathfrak{M} := \mathfrak{L}_1/\mathfrak{L}_2$ (resp. $\hat{\mathfrak{M}} := \hat{\mathfrak{L}}_1/\hat{\mathfrak{L}}_2$). Apparently, there are a φ-action and a $\hat{G}$-action on $\hat{\mathfrak{M}}$ induced from $\hat{\mathfrak{L}}_1$ and $\hat{\mathfrak{L}}_2$. We claim that $\hat{\mathfrak{M}} \simeq \widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ as φ-modules and $(\mathfrak{M}, \varphi, \hat{G})$ is a $(\varphi, \hat{G})$-modules. To see these, tensor $\widehat{\mathcal{R}}$ to the exact sequence $0 \to \mathfrak{L}_2 \to \mathfrak{L}_1 \to \mathfrak{M} \to 0$. By the proof of Lemma 3.1.2, we see that the sequence $0 \to \hat{\mathfrak{L}}_2 \to \hat{\mathfrak{L}}_1 \to \widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \to 0$ is still exact. Thus $\hat{\mathfrak{M}} \simeq \widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ as φ-modules. Moveover, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \hat{\mathfrak{L}}_2 \arrow[r] & \hat{\mathfrak{L}}_1 \arrow[r] & \widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \arrow[r] & 0 \\
0 \arrow[r] & \mathfrak{L}_2 \arrow[r] \arrow[u, hook] & \mathfrak{L}_1 \arrow[r] \arrow[u, hook] & \mathfrak{M} \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

So φ-action and $\hat{G}$-action on $\hat{\mathfrak{M}}$ commutates, H_K acts on $\mathfrak{M}$ (as $\varphi(\mathfrak{S})$-submodule in (3.1.2)) trivially, and $\hat{G}$ acts on $\hat{\mathfrak{M}}/I_+\hat{\mathfrak{M}}$ trivially. Thus $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G})$ is a $(\varphi, \hat{G})$-module. Finally, to see that $\hat{T}_n(\hat{\mathfrak{M}}) \simeq T$ as $\mathbb{Z}_p[G]$-modules, it suffices to show that $\hat{T}_n(\hat{\mathfrak{M}}) \simeq \hat{T}(\hat{\mathfrak{L}}_2)/\hat{T}(\hat{\mathfrak{L}}_1)$ and we reduce the proof to the following Lemma. ☐

Lemma 3.1.4. *Let $0 \to \hat{\mathfrak{L}}_2 \to \hat{\mathfrak{L}}_1 \to \hat{\mathfrak{M}} \to 0$ be an exact sequence of (φ, G)-modules with $\hat{\mathfrak{L}}_1$, $\hat{\mathfrak{L}}_2$ finite free and $\hat{\mathfrak{M}}$ annihilated by p^n. Then we have an exact sequence of $\mathbb{Z}_p[G]$-modules $0 \to \hat{T}(\hat{\mathfrak{L}}_1) \to \hat{T}(\hat{\mathfrak{L}}_2) \to \hat{T}_n(\hat{\mathfrak{M}}) \to 0$.*

Proof. Let m be an integer not less than n. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & p^m\hat{\mathfrak{L}}_2 \arrow[r] \arrow[d, hook] & p^m\hat{\mathfrak{L}}_1 \arrow[r] \arrow[d, hook] & \hat{\mathfrak{M}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \hat{\mathfrak{L}}_2 \arrow[r] & \hat{\mathfrak{L}}_1 \arrow[r] & \hat{\mathfrak{M}} \arrow[r] & 0
\end{tikzcd}$$

where the last vertical map is $p^m = 0$. By Snake lemma, we have an exact sequence

$$0 \to \hat{\mathfrak{M}} \to (\hat{\mathfrak{L}}_2)_m \to (\hat{\mathfrak{L}}_1)_m \to \hat{\mathfrak{M}} \to 0. \tag{3.1.4}$$

Then we get a sequence of $\mathbb{Z}_p[G]$-modules

$$0 \to \hat{T}_m(\hat{\mathfrak{M}}) \to \hat{T}_m((\hat{\mathfrak{L}}_1)_m) \to \hat{T}_m((\hat{\mathfrak{L}}_2)_m) \to \hat{T}_m(\hat{\mathfrak{M}}) \to 0. \tag{3.1.5}$$

Since $\hat{T}_m((\hat{\mathfrak{L}}_i)_m) \simeq (\hat{T}(\hat{\mathfrak{L}}_i))_m$ for $i = 1, 2$, it suffices to show that the above sequence is exact. But the underlying Kisin modules of the exact sequence (3.1.4) is exact. Since $T_{\mathfrak{S}}$ is exact, we get an exact sequence

$$0 \to T_{\mathfrak{S}_m}(\mathfrak{M}) \to T_{\mathfrak{S}_m}((\mathfrak{L}_1)_m) \to T_{\mathfrak{S}_m}((\mathfrak{L}_2)_m) \to T_{\mathfrak{S}_m}(\mathfrak{M}) \to 0.$$

Now the exactness of (3.1.5) follows from Theorem 3.1.3.(1). □

Remark 3.1.5. For a fixed $T \in \mathrm{Rep}^{\mathrm{ss},r}_{\mathrm{tor}}(G)$, it may exist two different $(\varphi, \hat{G})$-modules $\hat{\mathfrak{M}}$, $\hat{\mathfrak{M}}'$ such that $\hat{T}_n(\hat{\mathfrak{M}}) \simeq \hat{T}_n(\hat{\mathfrak{M}}') \simeq T$. The classical example of this is that $T = \mathbb{Z}/p\mathbb{Z}$ with the trivial G-action and $K = \mathbb{Q}_p(\zeta_p)$.

3.2. **G_s-action on $\hat{T}(\hat{\mathfrak{L}})$.** Let $T \in \mathrm{Rep}^{\mathrm{ss},r}_{\mathrm{tor}}(G)$ be a p^n-torsion representation, and $T \simeq \Lambda'/\Lambda$ where $\Lambda \subset \Lambda'$ are G-stable $\mathbb{Z}_p$-lattices in a semistable representation V with Hodge-Tate weights in $\{0, \ldots, r\}$. By Theorem 3.1.3, there exists $(\varphi, \hat{G})$-modules $\hat{\mathfrak{L}}' \hookrightarrow \hat{\mathfrak{L}}$ such that $\hat{T}(\hat{\mathfrak{L}}) \hookrightarrow \hat{T}(\hat{\mathfrak{L}}')$ corresponds to the injection $\Lambda \subset \Lambda'$ and $\hat{T}_n(\hat{\mathfrak{M}}) \simeq T$ where $\hat{\mathfrak{M}} := \hat{\mathfrak{L}}/\hat{\mathfrak{L}}'$. Now write $\mathfrak{L}$, $\mathfrak{L}'$, $\mathfrak{M}$ the underlying Kisin modules for $\hat{\mathfrak{L}}$, $\hat{\mathfrak{L}}'$, $\hat{\mathfrak{M}}$ respectively. Set $\mathcal{D} := S[1/p] \otimes_{\varphi,\mathfrak{S}} \mathfrak{L}$ and recall $\underline{\epsilon}(g) := \frac{g(\pi)}{\pi}$ for any $g \in G$. §3.2 of [21] explains that there exists an unique $W(k)$-linear differential operator $N : \mathcal{D} \to \mathcal{D}$ over $N(u) = -u$ such that G acts on $B^+_{\mathrm{cris}} \otimes_S \mathcal{D} \simeq B^+_{\mathrm{cris}} \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{L}}$ *via*

$$g(a \otimes x) = \sum_{i=0}^{\infty} g(a)\gamma_i(-\log([\underline{\epsilon}(g)])) \otimes N^i(x), \quad \text{for any } a \in B^+_{\mathrm{cris}},\ x \in \mathcal{D}. \tag{3.2.1}$$

In particular, recall $t := -\log([\underline{\epsilon}])$ with $\underline{\epsilon} = \underline{\epsilon}(\tau)$ and τ is a fixed generator in G_{p^∞}. For any $x \in \mathfrak{L}$, we have $\tau(x) = \sum\limits_{i=0}^{\infty} \gamma_i(t) \otimes N^i(x)$. Let $A \subset B^+_{\mathrm{cris}}$ be a φ-stable subring. Set

$$I^{[m]}A = \{a \in A | \varphi^n(a) \in A \cap \mathrm{Fil}^m B^+_{\mathrm{cris}},\ \text{for all } n \geq 0\}.$$

By proposition 5.1.3 in [13], $I^{[m]}W(R)$ is generated by $([\underline{\epsilon}] - 1)^m$ and $v_R(\underline{\epsilon} - 1) = \frac{ep}{p-1}$.

Now, define $s_2(c) := n - 1 + \log_p(\frac{(p-1)c}{e}) = s_1(c) + 1$. We have the following lemma:

Lemma 3.2.1. *For any $s > s_2(c)$, $g \in G_s$, and $x \in \mathfrak{L}$*

$$g(x) - x \in ([\mathfrak{a}_R^{>pc}] + p^n W(R)) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{L}}$$

where, in a slight abuse of notations, we still denote by $[\mathfrak{a}_R^{>c}]$ the ideal of $W(R)$ generated by all $[x]$ with $x \in \mathfrak{a}_R^{>c}$.

Proof. Note that the G-action on $\hat{\mathfrak{L}}$ factors through $\hat{G}$. So it suffices to consider the action of $\hat{G}_s$, which is the image of G_s in $\hat{G}$. By Lemma 5.1.2 of [22] applied to K_s, we see that $\hat{G}_s := G_{s,p^\infty} \rtimes H_K$, where $G_{s,p^\infty} = \mathrm{Gal}(K_{\infty,p^\infty}/K_{s,p^\infty})$. Note that H_K acts on $\mathfrak{L}$ trivially and G_{s,p^∞} is topologically generated by τ^{p^s}. Thus it suffices to prove the proposition for $g = \tau^{p^s}$. Writing

$$\tau^{p^s} - 1 = \sum_{i=1}^{p^s} \binom{p^s}{i} (\tau - 1)^i = \sum_{i=1}^{p^s} \frac{p^s}{i} \binom{p^s - 1}{i - 1} (\tau - 1)^i$$

we see that it is enough to show

$$(\tau - 1)^i(x) \in ([\mathfrak{a}_R^{>pc}] + p^{n-s+v_p(i)} W(R)) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{L}}$$

for all integer i such that $v_p(i) > s - n$, *i.e.* $v_p(i) \geq s - n + 1$.

Using formula (3.2.1), an easy induction on l shows that

$$(\tau - 1)^l(x) = \sum_{m=l}^{\infty} \left(\sum_{i_1+\cdots+i_l=m, i_j \geq 1} \frac{m!}{i_1! \cdots i_l!} \right) \gamma_m(t) \otimes N^m(x) \tag{3.2.2}$$

for any $l \geq 0$ and $x \in \mathcal{D}$. In particular, $(\tau-1)^l(x) \in (I^{[l]}B^+_{\mathrm{cris}})(B^+_{\mathrm{cris}} \otimes_S \mathcal{D})$. Since $x \in \mathfrak{L}$ and $W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{L}}$ is G-stable, we get $(\tau-1)^l(x) \in (I^{[l]}W(R))(W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{L}})$. So it suffices to show that $([\underline{\epsilon}]-1)^i \in [\mathfrak{a}_R^{>pc}] + p^{n-s+v_p(i)}W(R)$ for any i satisfying $v_p(i) \geq s-n+1$. Write $i = p^{v+s-n+1}m$ with $v \geq 0$ and $p \nmid m$. From $v_R(\underline{\epsilon}-1) = \frac{ep}{p-1}$, it follows that $([\underline{\epsilon}]-1)^{p^{s-n+1}} \in [\mathfrak{a}_R^{>pc}] + pW(R)$. By induction (on v), we easily find $([\underline{\epsilon}]-1)^{p^{v+s-n+1}} \in [\mathfrak{a}_R^{>pc}] + p^{v+1}W(R)$, which is exactly the expected result. □

3.3. **Comparison between $\hat{J}_{n,c}(\hat{\mathfrak{M}})$ and $J_{n,c}(\mathfrak{M})$.** Let T be a p^n-torsion semistable representation and $\hat{\mathfrak{M}}$ an attached $(\varphi, \hat{G})$-module *via* Theorem 3.1.3.(3). We have the following definitions and results similar to (2.3.1). For any non negative real number c, set

$$\hat{J}_{n,c}(\hat{\mathfrak{M}}) := \mathrm{Hom}_{\widehat{\mathcal{R}}_n,\varphi}(\hat{\mathfrak{M}}, W_n(R)/[\mathfrak{a}_R^{>pc}]).$$

and $\hat{J}_{n,\infty}(\hat{\mathfrak{M}}) = \hat{T}_n(\hat{\mathfrak{M}})$.

For any $c \leq \infty$, $\hat{J}_{n,c}(\hat{\mathfrak{M}})$ is a $\mathbb{Z}_p[G]$-module and, for any $c \leq c' \leq \infty$, the canonical projection induces a map $\hat{\rho}_{c',c} : \hat{J}_{n,c'}(\hat{\mathfrak{M}}) \to \hat{J}_{n,c}(\hat{\mathfrak{M}})$. Moreover, to each $f \in J_{n,c}(\mathfrak{M})$, one can attach a morphism $\theta_{n,c}(f) \in \hat{J}_{n,c}(\hat{\mathfrak{M}})$ defined by:

$$(\forall \alpha \in \widehat{\mathcal{R}})\,(\forall x \in \mathfrak{M}) \quad \theta_{n,c}(f)(\alpha \otimes x) = \alpha\varphi(f(x)). \tag{3.3.1}$$

Proposition 3.3.1. *For any nonnegative integer $s > s_2(c) = n-1+\log_p(\frac{(p-1)c}{e})$, $\theta_{n,c} : J_{n,c}(\mathfrak{M}) \to \hat{J}_{n,c}(\hat{\mathfrak{M}})$ is an isomorphism of $\mathbb{Z}_p[G_s]$-modules.*

Remark 3.3.2. Since $s_2(c) = s_1(c)+1 \geq s_1(c)$, Proposition 2.5.3 shows that $J_{n,c}(\mathfrak{M})$ is endowed with an action of G_s. Hence it makes sense to claim that $\theta_{n,c}$ is G_s-equivariant.

Proof. It is routine to check that $\theta_{n,c}(f)$ is well defined and preserves Frobenius. Hence $\theta_{n,c}$ is also well defined. Let's first prove that it is bijective. Remark that $\varphi : W_n(R)/[\mathfrak{a}_R^{>c}] \xrightarrow{\sim} W_n(R)/[\mathfrak{a}_R^{>pc}]$ is an isomorphism. It follows easily that $\theta_{n,c}$ is injective. For any $\hat{f} \in \hat{J}_{n,c}(\hat{\mathfrak{M}})$, set $f' := \hat{f}|_{\mathfrak{M}}$ (recall that we regard $\mathfrak{M}$ as a $\varphi(\mathfrak{S})$-submodule of $\hat{\mathfrak{M}}$ via (3.1.2)). Then $f' : \mathfrak{M} \to W_n(R)/[\mathfrak{a}_R^{>pc}]$ is $\varphi(\mathfrak{S})$-linear map and is compatible with Frobenius. Since $\varphi : W_n(R)/[\mathfrak{a}_R^{>c}] \simeq W_n(R)/[\mathfrak{a}_R^{>pc}]$, we can set $f = \varphi^{-1}(f') : \mathfrak{M} \to W_n(R)/[\mathfrak{a}_R^{>c}]$. It is finally easy to check that f belongs to $J_{n,c}(\mathfrak{M})$ and that $\theta_{n,c}(f) = \hat{f}$. Hence $\theta_{n,c}$ is surjective, as required.

It remains to prove that $\theta_{n,c}$ is G_s-equivariant. Let $g \in G_s$, $\alpha \in \widehat{\mathcal{R}}$ and $x \in \mathfrak{M}$. Expanding the definitions, we get $g(\theta_{n,c}(f))(\alpha \otimes x) = \alpha g(\theta_{n,c}(f)(g^{-1}(1 \otimes x)))$. Moreover Lemma 3.2.1 shows that $g^{-1}(1 \otimes x)$ is congruent to $1 \otimes x$ modulo $[\mathfrak{a}_R^{>pc}]$ and hence that these two terms have same image under $\theta_{n,c}(f)$. Thus:

$$\begin{aligned} g(\theta_{n,c}(f))(\alpha \otimes x) &= \alpha g(\theta_{n,c}(f)(g^{-1}(1 \otimes x))) = \alpha g(\theta_{n,c}(f)(1 \otimes x)) \\ &= \alpha g(\varphi(f(x))) = \alpha\varphi(g(f(x))) = \theta_{n,c}(g(f))(\alpha \otimes x) \end{aligned}$$

and equivariance is proved. □

Recall that we have fixed an integer N such that $u^N = 0$ in $W[u]/E(u)^r$ and defined $b = \frac{N}{p-1}$ and $a = \frac{pN}{p-1}$. Combining Propositions 2.3.3 and 3.3.1, we directly get the following.

Corollary 3.3.3. *The morphism $\hat{\rho}_{\infty,b} : \hat{T}_n(\hat{\mathfrak{M}}) \to \hat{J}_{n,b}(\hat{\mathfrak{M}})$ is injective and its image is $\hat{\rho}_{a,b}(\hat{J}_{n,a}(\hat{\mathfrak{M}}))$.*

Theorem 3.3.4. *With previous notations, the map $\theta_n : T_{\mathfrak{S}_n}(\mathfrak{M}) \xrightarrow{\sim} \hat{T}_n(\hat{\mathfrak{M}}) \simeq T$ is an isomorphism of $\mathbb{Z}_p[G_s]$-modules for all integer $s > s_{\min} = n-1+\log_p(\frac{N}{e})$.*

Proof. We already know that θ_n is bijective (Theorem 3.1.3.(1)). Now, consider the following commutative diagram:

$$\begin{tikzcd} T_{\mathfrak{S}_n}(\mathfrak{M}) \arrow[rr, "\theta_n", "\sim"'] \arrow[d, "\rho_{\infty,a}"'] & & \hat{T}_n(\hat{\mathfrak{M}}) \arrow[dd, hook, "\hat{\rho}_{\infty,b}"] \\ J_{n,a}(\mathfrak{M}) \arrow[d, "\rho_{a,b}"'] & & \\ J_{n,b}(\mathfrak{M}) \arrow[rr, "\theta_{n,b}", "\sim"'] & & \hat{J}_{n,b}(\hat{\mathfrak{M}}) \end{tikzcd}$$

Note that $s_{\min} = s_1(a) = s_2(b)$. Thus by definition of G_s-action on $T_{\mathfrak{S}_n}(\mathfrak{M})$ (resp. by Proposition 2.5.3, resp. by Proposition 3.3.1), $\rho_{\infty,a}$ (resp. $\rho_{a,b}$, resp. $\theta_{n,b}$) is G_s-equivariant. Since $\hat{\rho}_{\infty,b}$ is injective (Corollary 3.3.3) and G_s-equivariant, we deduce that θ_n is also G_s-equivariant as claimed. □

We end this section by giving a proof of Theorem 1.3 of introduction. For convenience of the reader, we first recall its statement:

Corollary 3.3.5 (Theorem 1.3)**.** *Let V and V' be two semistable representations of G with Hodge-Tate weights in $\{0,\dots,r\}$. Let T (resp. T') a quotient of two G-stable lattices in V (resp. V') which is annihilated by p^n. Then any G_∞-equivariant morphism $f : T \to T'$ is G_s-equivariant for all integer $s > n-1+\log_p(nr)$.*

Proof. Consider $\mathfrak{M}$ (resp. $\mathfrak{M}'$) some Kisin module such that $T_{\mathfrak{S}_n}(\mathfrak{M}) = T$ (resp. $T_{\mathfrak{S}_n}(\mathfrak{M}') = T'$). We may assume that $\mathfrak{M}$ and $\mathfrak{M}'$ are maximal in the sense of [8]. Then by Corollary 3.3.10 in *loc. cit.*, f comes from a morphism $g : \mathfrak{M}' \to \mathfrak{M}$. Using Theorem 3.3.4, one easily see that $T_{\mathfrak{S}_n}(g) = f$ is G_s-equivariant. □

4. Ramification bound

In this section, we give proofs of Theorem 1.1 based on above preparations. Our strategy is similar to those in [1], [10], [15] and [16]. Let n be a positive integer. Recall that we have defined several constants, that are:

- N is an integer such that $u^N = 0$ in $W_n[u]/E(u)^r$ (recall also that one may choose $N = ern$);
- $b = \frac{N}{p-1}$ and $a = \frac{pN}{p-1}$;
- $s_0(a) = n-1+\log_p(\frac{a}{e}) = n + \log_p(\frac{N}{e(p-1)})$;
- $s_{\min} = s_1(a) = s_2(b) = n-1+\log_p(\frac{N}{e})$.

Note that if we have chosen $N = ern$, then $s_0(a)$ is nothing but the minority of s that appears in Theorem 1.4. Let $T = \Lambda/\Lambda'$ be a quotient of two lattices in a semistable representation and assume that T is annihilated by p^n. Since we have a surjective map $\Lambda/p^n\Lambda \to T$, it is enough to bound ramification for $\Lambda/p^n\Lambda$. Hence, without loss of generality, we may assume that T is free over $\mathbb{Z}/p^n\mathbb{Z}$. By Theorem 3.1.3, there exists a $(\varphi,\hat{G})$-module $\hat{\mathfrak{M}}$ such that $\hat{T}_n(\hat{\mathfrak{M}}) = T$. With our extra assumption, $\mathfrak{M}$ is finite free over $\mathfrak{S}_n$.

From now on, we fix an integer $s > s_0(a)$. Remark that $s_0(a) > s_{\min}$ so that we also have $s > s_{\min}$. Hence theory developed in previous sections applies. In particular, by Propositions 2.5.2 and 2.5.3, for all $c \in [0, ep^{s-n+1}[$, we have a G_s-equivariant isomorphism

$$J_{n,c}(\mathfrak{M}) \simeq J^{(s)}_{n,c}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S},\varphi}\left(\mathfrak{M}, W_n(k) \otimes_{W_n(k),\sigma^s} \frac{W_n(\mathcal{O}_{\bar{K}}/p)}{[\mathfrak{a}_{\bar{K}}^{>c/p^s}]}\right) \tag{4.0.2}$$

where the structure of $\mathfrak{S}$-module on $W_n(\mathcal{O}_{\bar{K}}/p)$ is given by $u \mapsto 1 \otimes \pi_s$. Moreover, by Corollary 3.3.3 and Theorem 3.3.4

$$T|_{G_s} \simeq \mathrm{im}\, \rho_{a,b} : J_{n,a}(\mathfrak{M}) \to J_{n,b}(\mathfrak{M}). \tag{4.0.3}$$

Denote by L the splitting field of T, that is, $L = (\bar{K})^{\mathrm{Ker}(\rho)}$, where $\rho : G_K \to \mathrm{GL}_{\mathbb{Z}_p}(T)$ the attached group homomorphism. Set $L_s = K_sL$.

4.1. **The sets $J_{n,c}^{(s),E}(\mathfrak{M})$.** Let E be an algebraic extension of K_s inside $\bar{K}$. By restriction, the valuation v_K induces a valuation on E and one may define, for all nonnegative real number c, $\mathfrak{a}_E^{\geq c} = \{x \in E \,/\, v_K(x) \geq c\}$ and $\mathfrak{a}_E^{>c} = \{x \in E \,/\, v_K(x) > c\}$. If c belongs to the interval $[0, ep^{s-n+1}[$, we put

$$J_{n,c}^{(s),E}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S},\varphi}\left(\mathfrak{M}, W_n(k) \otimes_{W_n(k),\sigma^s} \frac{W_n(\mathcal{O}_E/p)}{[\mathfrak{a}_E^{>c/p^s}]}\right).$$

They are $\mathbb{Z}_p$-modules, and if E/K is Galois, they are endowed with an action of G_s. As usual, if $0 \leq c \leq c' \leq ep^{s-n+1}$, we have a natural morphism $\rho_{c',c}^{(s),E} : J_{n,c'}^{(s),E}(\mathfrak{M}) \to J_{n,c}^{(s),E}(\mathfrak{M})$. Apparently $J_{n,c}^{(s),E}(\mathfrak{M})$ injects $J_{n,c}^{(s),\bar{K}}(\mathfrak{M}) = J_{n,c}^{(s)}(\mathfrak{M})$.

The aim of this subsection is to show the following theorem.

Theorem 4.1.1. *Notations as above. The natural injection $\rho_{a,b}^{(s),E}(J_{n,a}^{(s),E}(\mathfrak{M})) \subset \rho_{a,b}(J_{n,a}(\mathfrak{M}))$ is bijective if and only if $L_s \subset E$.*

Remark 4.1.2. By (4.0.3), $\rho_{a,b}(J_{n,a}(\mathfrak{M}))$ is canonically isomorphic to T as a $\mathbb{Z}_p[G_s]$-module.

In order to achieve the proof, we will need to lift $J_{n,c}^{(s),E}(\mathfrak{M})$ at $\mathcal{O}_E$-level. We begin by defining a map $\varphi : W_n(\mathcal{O}_E) \to W_n(\mathcal{O}_E)$ by

$$\varphi(z) := (z_0^p, \ldots, z_{n-1}^p).$$

Note that φ is *not* a ring homomorphism. Nevertheless one easily check that $\varphi([\lambda]z) = [\lambda^p]\varphi(z)$ for $\lambda \in \mathcal{O}_E$ and $z \in W_n(\mathcal{O}_E)$ and φ is G_s-equivariant.

Remark 4.1.3. If A is any ring, one can always define Frobenius $\phi : W(A) \to W(A)$ by $w_m(\phi(x)) = w_{m+1}(x)$, $\forall x \in W(A)$, where $w_n(x)$ is the m-th ghost component of x. Then ϕ can be proved to be a ring homomorphism (see p.14 in [17]). Unfortunately, such Frobenius does not preserve the kernel of natural projection $W(A) \to W_n(A)$ unless A has characteristic p. Hence it is not well-defined on $W_n(A)$ if A has characteristic 0.

Recall now that we have assumed that $\mathfrak{M}$ is finite $\mathfrak{S}_n$-free. Select a basis $(e_1, \ldots, e_d)$ of $\mathfrak{M}$ and write $\varphi(e_1, \ldots, e_d) = (e_1, \ldots, e_d)A$ with $A \in \mathrm{M}_d(\mathfrak{S}_n)$. As discussed in §2.3, there exists $B \in \mathrm{M}_d(\mathfrak{S}_n)$ such that $AB = u^N I$. Let $\tilde{A}$ and $\tilde{B}$ be matrices in $\mathrm{M}_d(W_n(\mathcal{O}_{K_s}))$ that respectively lifts images of A and B under the ring homomorphism $\mathfrak{S}_n \to W_n(\mathcal{O}_{K_s}/p)$, $u \mapsto \pi_s$, $\lambda \mapsto \sigma^{-s}(\lambda)$ ($\lambda \in W_n(k)$). Apparently, $\tilde{A}\tilde{B} \equiv [\pi_s]^N \pmod{W_n(p\mathcal{O}_{K_s})}$. Thus, there exists a matrix R with coefficients in $W_n([\mathfrak{a}_{K_s}^{\geq 0}])$ such that $\tilde{A}\tilde{B} = [\pi_s]^N(I+R)$ (where I is the identity matrix). Noting that $I + R$ is invertible, one get $\tilde{A}\tilde{B}(I+R)^{-1} = [\pi_s]^N I$. Hence, up to replacing $\tilde{B}$ by $\tilde{B}(I+R)^{-1}$, one may assume that $\tilde{A}\tilde{B} = [\pi_s]^N I$. Finally define a set

$$\tilde{J}_n^{(s),E}(\mathfrak{M}) := \left\{(\tilde{x}_1, \ldots, \tilde{x}_d) \in W_n(\mathcal{O}_E)^d \,/\, (\varphi(\tilde{x}_1), \ldots, \varphi(\tilde{x}_d)) = (\tilde{x}_1, \ldots, \tilde{x}_d)\tilde{A}\right\}.$$

The natural projection $W_n(\mathcal{O}_E) \to W_n(\mathcal{O}_E/p) \to W_n(\mathcal{O}_E/p)/[\mathfrak{a}_E^{>c/p^s}]$ induces a map $\tilde{\rho}_c^{(s),E} : \tilde{J}_n^{(s),E}(\mathfrak{M}) \to J_{n,c}^{(s),E}(\mathfrak{M})$.

Lemma 4.1.4. *$\tilde{\rho}_b^{(s),E}$ is injective and its image is $\rho_{a,b}^{(s),E}(J_{n,a}^{(s),E}(\mathfrak{M}))$.*

Proof. During the proof, if z is any element in $W_n(\mathcal{O}_E)$, we will denote by $z^{(i)} \in \mathcal{O}_E$ its i-th component. By the same way, we define $Z^{(i)}$ for a matrix Z with entries in $W_n(\mathcal{O}_E)$. Also, if Z is a matrix with entries in $\mathcal{O}_E$, we will denote by $v_K(Z)$ the smallest valuation of coefficients of Z.

We first show $\tilde{\rho}_b^{(s),E}$ is an injection. Assume that X and Y are in $\tilde{J}_n^{(s),E}(\mathfrak{M})$ such that $\tilde{\rho}_b^{(s),E}(X) - \tilde{\rho}_b^{(s),E}(Y) = 0$. Then $Z = X - Y \in [\mathfrak{a}_E^{>b/p^s}] + W_n(p\mathcal{O}_E) = [\mathfrak{a}_E^{>b/p^s}]$. We need to prove that $Z = 0$. Assume by contradiction that it is false and consider m the smallest number such that $Z^{(m)} \neq 0$. Define $W := Z\tilde{A} = \varphi(X) - \varphi(Y) = \varphi(Y+Z) - \varphi(Y)$. Easy computations show that $W^{(i)} = 0$ for $i < m$ and

$$W^{(m)} = \sum_{i=1}^{p} \binom{p}{i} (Y^{(m)})^{p-i}(Z^{(m)})^i. \tag{4.1.1}$$

where the multiplication is computed component by component. If $1 \leq i < p$, we have

$$v_K\left(\binom{p}{i}(Y^{(m)})^{p-i}(Z^{(m)})^i\right) \geq e + v_K(Z^{(m)}) > Np^{m-1-s} + v_K(Z^{(m)})$$

and, using $v_K(Z^{(m)}) > bp^{m-1-s}$ (recall that $Z \in [\mathfrak{a}_E^{>b/p^s}]$), we find

$$v_K((Z^{(m)})^p) > (p-1)bp^{m-1-s} + v_K(Z^{(m)}) = Np^{m-1-s} + v_K(Z^{(m)}).$$

Hence each term in RHS of (4.1.1) has valuation greater than $Np^{m-1-s}+v_K(Z^{(m)})$. So $v_K(W^{(m)}) > Np^{m-1-s} + v_K(Z^{(m)})$. But, on the other hand, comparing the m-th component of $W\tilde{B} = [\pi_s]^N Z$, we get $v_K(W^{(m)}) \leq Np^{m-1-s} + v_K(Z^{(m)})$. This is a contradiction and injectivity follows.

Let us now prove the second statement. Remark first that for all $c \in [0, ep^{s-n+1}[$, we have $W_n(\mathcal{O}_E/p)/[\mathfrak{a}_E^{>c/p^s}] \simeq W_n(\mathcal{O}_E)/[\mathfrak{a}_E^{>c/p^s}]$ and hence that $J_{n,c}^{(s),E}(\mathfrak{M})$ can be identified with

$$\left\{(\tilde{x}_1, \ldots, \tilde{x}_d) \in W_n(\mathcal{O}_E)^d \,/\, (\varphi(\tilde{x}_1), \ldots, \varphi(\tilde{x}_d)) \equiv (\tilde{x}_1, \ldots, \tilde{x}_d)\tilde{A} \pmod{[\mathfrak{a}_E^{>c/p^s}]}\right\}$$

modulo $[\mathfrak{a}_E^{>c/p^s}]^d$. Let $X = (\tilde{x}_1, \ldots, \tilde{x}_d) \in W_n(\mathcal{O}_E)^d$ be an solution as above. We have equation $\varphi(X) = X\tilde{A} + Q'$ with coefficients of Q' in $[\mathfrak{a}_E^{>a/p^s}]$. Actually, the congruence holds in $[\mathfrak{a}_E^{\geqslant a'}]$ for some a' satisfying $\frac{e}{p^{n-1}} \geq a' > \frac{a}{p^s}$. Note that $\frac{e}{p^{n-1}} \geq a'$ implies that $W_n(p\mathcal{O}_E) \subset [\mathfrak{a}_E^{\geqslant a'}]$. For the rest of the proof, let $\alpha \in \mathcal{O}_E$ be an element of valuation a'. By the similar argument as in Lemma 2.3.1.(2), $[\mathfrak{a}_E^{\geqslant a'}]$ is the principal ideal generated by $[\alpha]$. Therefore, one have $\varphi(X) - X\tilde{A} = [\alpha]Q$ with the coefficients of Q in $W_n(\mathcal{O}_E)$. We want to prove that there exists a matrix Y with coefficients in $[\mathfrak{a}_E^{>b/p^s}]$ such that $(X+Y)\tilde{A} = \varphi(X+Y)$. Let us search Y of the shape $[\beta]Z$ with $\beta = \frac{\alpha}{\pi_s^N}$ (which belongs to $\mathcal{O}_E$ because of valuations) and coefficients of Z in $W_n(\mathcal{O}_E)$. Our condition then becomes $(X+[\beta]Z)\tilde{A} = \varphi(X+[\beta]Z)$. Multiplying $\tilde{B}$ on both sides and noting that $[\pi_s]$ is a non-zero divisor of $W_n(\mathcal{O}_E)$, we need to prove that the following equation has a (necessarily unique) solution:

$$[\pi_s]^N X + [\pi_s]^N [\beta] Z = \varphi(X + [\beta]Z)\tilde{B}. \tag{4.1.2}$$

Let us prove by induction on n. If $n = 1$, set $Z_0 = 0$ and $Z_{l+1} = \pi_s^{-N}\beta^{-1}(\varphi(X+\beta Z_l)\tilde{B} - \pi_s^N X)$. To see that Z_{l+1} is in $\mathcal{O}_E$, note that

$$\begin{aligned}\varphi(X+\beta Z_l)\tilde{B} - \pi_s^N X &= (X+\beta Z_l)^p\tilde{B} - \pi_s^N X \\ &= (\varphi(X)\tilde{B} - \pi_s^N X) + \sum_{i=1}^{p-1}\tbinom{p}{i}X^{p-i}(\beta Z_l)^i\tilde{B} + \beta^p(Z_l)^p\tilde{B}.\end{aligned}$$

Since $\varphi(X)\tilde{B} - \pi_s^N X = \alpha Q\tilde{B}$, $v_K(\pi_s^N\beta) \leq v_K(\alpha) \leq v_K(p)$ and $(p-1)v_K(\beta) \geq v_K(\pi_s^N)$, we see that Z_{l+1} is in $\mathcal{O}_E$. Note that

$$\begin{aligned} Z_{l+1} - Z_l &= \pi_s^{-N}\beta^{-1}(\varphi(X+\beta Z_l) - \varphi(X+\beta Z_{l-1}))\tilde{B} \\ &= \pi_s^{-N}\beta^{-1}\sum_{i=1}^{p}\binom{p}{i}X^{p-i}\beta^i(Z_l^i - Z_{l-1}^i)\tilde{B}.\end{aligned}$$

Since $v_K(p) \geq v_K(\pi_s^N\beta)$ and $(p-1)v_K(\beta) > v_K(\pi_s^N)$, we see that $v_K(Z_{l+1}-Z_l) \geq \gamma + v_K(Z_l - Z_{l-1})$, where $\gamma = \min(v_K(\beta), v_K(\beta^{p-1}\pi_s^{-N})) > 0$. Hence Z_l converge to Z and we solve the equation (4.1.2) for $n = 1$.

Now assume that equation (4.1.2) has a solution for $n \leq m-1$, consider the $n = m$ case. Recall that $z^{(i)} \in \mathcal{O}_E$ represents the i-th component of $z \in W_m(\mathcal{O}_E)$. Set $Z_0 = (Z_0^{(0)}, \ldots, Z_0^{(m-1)})$ where $Z_0^{(m-1)} = 0$ and $(Z_0^{(0)}, \ldots, Z_0^{(m-2)})$ is the solution of (4.1.2) in $n = m-1$ case. Now set $Z_{l+1} = [\pi_s]^{-N}[\beta]^{-1}(\varphi(X+[\beta]Z_l)\tilde{B} - [\pi_s]^N X)$. Since $(Z_0^{(0)}, \ldots, Z_0^{(m-2)})$ is the solution of (4.1.2) in $n = m-1$ case, we see that $Z_l^{(i)} = Z_{l+1}^{(i)}$ for all l and $i = 0, \ldots, m-2$. Now it suffices to check that Z_{l+1} has coefficients in $W_m(\mathcal{O}_E)$ and Z_l converges.

Since $\varphi(X+[\beta]Z_l) = \varphi(X)+\varphi([\beta]Z_l)$ in $W_m(\mathcal{O}_E/p)$, we have $\varphi(X+[\beta]Z_l) = \varphi(X)+\varphi([\beta]Z_l)+C'$ with coefficients of C' in $W_m(p\mathcal{O}_E)$. Since $W_m(p\mathcal{O}_E) \subset [\alpha]W_m(\mathcal{O}_E)$, we can write $C' = [\alpha]C$ with coefficients of C in $W_m(\mathcal{O}_E)$. Hence $\varphi(X+[\beta]Z_l)\tilde{B} - [\pi_s]^N X = (\varphi(X)\tilde{B} - [\pi_s]^N X) + [\beta]^p\varphi(Z_l)\tilde{B} +$

$[\alpha]C\tilde{B} = [\alpha]Q\tilde{B} + [\beta]^p\varphi(Z_l)\tilde{B} + [\alpha]C\tilde{B}$. Since $(p-1)v_K(\beta) > v_K(\pi_s^N)$, we see that Z_{l+1} is well defined. Now $[\pi_s]^N[\beta](Z_{l+1} - Z_l) = (\varphi(X + [\beta]Z_l) - \varphi(X + [\beta]Z_{l-1}))\tilde{B} = W\tilde{B}$, where

$$\begin{array}{rcl} W & = & \varphi(X + [\beta]Z_l) - \varphi(X + [\beta]Z_{l-1}) \\ & = & \varphi(X + [\beta]Z_{l-1} + [\beta](Z_l - Z_{l-1})) - \varphi(X + [\beta]Z_{l-1}) \\ & = & \varphi(V + [\beta](Z_l - Z_{l-1})) - \varphi(V) \end{array}$$

with $V = X + [\beta]Z_{l-1}$. Since $Z_l^{(i)} = Z_{l+1}^{(i)}$ for all l and $0 \le i \le m-2$, $W^{(i)} = 0$ for $i = 0, \dots, m-2$ and

$$W^{(m-1)} = \sum_{i=1}^{p} \binom{p}{i} (V^{(m-1)})^{p-i} (\beta^{p^{m-1}}(Z_l^{(m-1)} - Z_{l-1}^{(m-1)}))^i.$$

Hence $v_K((\pi_s^N\beta)^{p^{m-1}}) + v_K(Z_{l+1}^{(m-1)} - Z_l^{(m-1)}) \ge v_K(\beta^{p^m}) + v_K(Z_l^{(m-1)} - Z_{l-1}^{(m-1)})$ if $i = p$ and

$$v_K((\pi_s^N\beta)^{p^{m-1}}) + v_K(Z_{l+1}^{(m-1)} - Z_l^{(m-1)})) \ge v_K(p) + v_K(\beta) + v_K(Z_l^{(m-1)} - Z_{l-1}^{(m-1)})$$

if $1 \le i \le p-1$. Since $(p-1)v_K(\beta) > v_K(\pi_s^N)$ and $v_K((\pi_s^N\beta)^{p^{m-1}}) \le v_K(p)$, we get

$$v_K(Z_{l+1}^{(m-1)} - Z_l^{(m-1)})) \ge \gamma + v_K(Z_l^{(m-1)} - Z_{l-1}^{(m-1)}),$$

where $\gamma = \min(v_K(\beta), v_K(\beta^{p-1}\pi_s^{-N}))$. Hence Z_l converges and we are done. □

Proof of Theorem 4.1.1. We have G_s-equivariant bijections of sets:

$$\begin{array}{rcll} \tilde{J}_n^{(s),\bar{K}}(\mathfrak{M}) & \simeq & \rho_{a,b}^{(s)}(J_{n,b}^{(s)}(\mathfrak{M})) & \text{by Lemma 4.1.4 applied with } E = \bar{K} \\ & \simeq & \rho_{a,b}(J_{n,b}(\mathfrak{M})) & \text{by Formula (4.0.2)} \\ & \simeq & T|_{G_s} & \text{by Proposition 2.3.3 and Theorem 3.3.4.} \end{array}$$

Taking fixed points under $\mathrm{Gal}(\bar{K}/E)$, we get a natural bijection $\tilde{J}_n^{(s),E}(\mathfrak{M}) \simeq T^{\mathrm{Gal}(\bar{K}/E)}$. Hence again by Lemma 4.1.4, $T^{\mathrm{Gal}(\bar{K}/E)} \simeq \rho_{a,b}^{(s),E}(J_{n,a}^{(s),E}(\mathfrak{M}))$, from what the theorem is easily deduced. □

4.2. **Proof of Theorem 1.4.** Recall that $L_s = K_sL$ with L the splitting field of T. We are now ready to bound the ramification of L_s. To do this, we need to recall the property $(P_m^{F/N})$ described by Fontaine (Proposition 1.5, [10]). But before that, in order to fix notations, we would like to recall some definitions about ramification filtration.

Let F_1/F_0 be a Galois extension of p-adic fields, with Galois group G. For all nonnegative real number λ, we define a normal subgroup $G_{(\lambda)}$ of G by

$$G_{(\lambda)} = \{\sigma \in G \,/\, v_{F_1}(\sigma(x) - x) \ge \lambda,\, \forall x \in \mathcal{O}_{F_1}\}$$

where v_{F_1} is the valuation normalized by $v_{F_1}(F_1^\star) = \mathbb{Z}$ and $\mathcal{O}_{F_1}$ is the ring of integers of F_1. We underline that we use here conventions of [10] and that they differ by a shift with conventions of [24], Chap. IV. By definition $G_{(\lambda)}$ is called the *lower ramification filtration* of G. Now, let $\varphi_{F_1/F_0} : [0, +\infty[\to [0, +\infty[$ be the function defined by

$$\varphi_{F_1/F_0}(\lambda) := \int_0^\lambda \frac{\mathrm{Card}\, G_{(t)}}{\mathrm{Card}\, G_{(1)}} dt.$$

It is increasing, continuous, concave, piecewise affine and bijective. Let ψ_{F_1/F_0} denote its inverse and set $G^{(\mu)} = G_{(\psi_{F_1/F_0}(\mu))}$: it is the *upper ramification filtration.* Finally call λ_{F_1/F_0} (resp. μ_{F_1/F_0}) the last break in the lower (resp. upper) ramification filtration of G, that is the infimum of λ (resp. μ) such that $G_{(\lambda)} = 1$ (resp. $G^{(\mu)} = 1$). Obviously $\mu_{F_1/F_0} = \varphi_{F_1/F_0}(\lambda_{F_1/F_0})$.

We refer to [24], Chap. IV for basic properties of these filtrations, and especially for Herbrand's theorem that allows us to extend upper ramification filtration to infinite algebraic extension. In particular, G_K is itself filtered by normal closed subgroups $G_K^{(\mu)}$. Note that $\mu_{F_1/F_0} = \inf\{\mu \in \mathbb{R}^+ \,/\, G_{F_0}^{(\mu)} \subset G_{F_1}\}$ where G_{F_0} and G_{F_1} denote the absolute Galois groups of F_0 and F_1 respectively.

Proposition 4.2.1 (M. Yoshida)**.** *Let F_1 and F_0 be finite extensions of K with $F_0 \subset F_1 \subset \bar{K}$ and F_1 is Galois. For any positive real number m, consider the following property*

$$(P_m^{F_1/F_0}) : \begin{cases} \text{For any algebraic extension } E \text{ over } F_0, \\ \text{if there exists an } \mathcal{O}_{F_0}\text{-algebrahomomorphism } \mathcal{O}_{F_1} \to \mathcal{O}_E/\mathfrak{a}_E^{\geq m}, \\ \text{then there exists a } F_0\text{-injection } F_1 \hookrightarrow E. \end{cases}$$

Let $e_{F_0/K}$ denote the ramification index of F_0/K. Then

$$\frac{\mu_{F_1/F_0}}{e_{F_0/K}} = \inf\{m \in \mathbb{R}^+ \,/\, \text{the property } (P_m^{F_1/F_0}) \text{ holds}\}.$$

Proof. See Proposition 3.4 in [27]. □

We will also need the following corollary:

Corollary 4.2.2. *If $(P_m^{F_1/F_0})$ holds for a positive real number m, then $v_K(\mathcal{D}_{F_1/F_0}) < m$.*

Proof. If F_1/F_0 is unramified, $v_K(\mathcal{D}_{F_1/F_0}) = 0$ and the corollary is obvious. If not, Proposition 1.3 of [10] shows that $e_{F_0/K} v_K(\mathcal{D}_{F_1/F_0}) < \mu_{F_1/F_0}$, and we are done. □

We claim that $(P_m^{L_s/K_s})$ holds for $m = ap^{n-1-s}$. To see this, pick $f : \mathcal{O}_{L_s} \to \mathcal{O}_E/\mathfrak{a}_E^{\geq m}$ an $\mathcal{O}_{K_s}$-algebra homomorphism. Obviously, for any real number $c \in [0, m]$, f induces a map $f_c : \mathcal{O}_{L_s}/\mathfrak{a}_{L_s}^{\geq c} \to \mathcal{O}_E/\mathfrak{a}_E^{\geq c}$.

Lemma 4.2.3. (1) *For any $c \leq m$, f_c is injective.*
(2) *For any $c \leq a$, $f_{cp^{n-1-s}}$ induces an injection*

$$W_n(\mathcal{O}_{L_s}/p)/[\mathfrak{a}_{L_s}^{\geq c/p^s}] \hookrightarrow W_n(\mathcal{O}_E/p)/[\mathfrak{a}_E^{\geq c/p^s}].$$

Proof. (1) It is the same as the proof of Lemma 4.4 of [15].

(2) Using an analogue of Lemma 2.3.1.(1), one easily proves that natural projections $\mathcal{O}_{L_s}/p \to \mathcal{O}_{L_s}/\mathfrak{a}_{L_s}^{\geq cp^{n-1-s}}$ and $\mathcal{O}_E/p \to \mathcal{O}_E/\mathfrak{a}_E^{\geq cp^{n-1-s}}$ induce isomorphisms

$$\begin{aligned} W_n(\mathcal{O}_{L_s}/p)/[\mathfrak{a}_{L_s}^{\geq c/p^s}] &\simeq W_n(\mathcal{O}_{L_s}/\mathfrak{a}_{L_s}^{\geq cp^{n-1-s}})/[\mathfrak{a}_{L_s}^{\geq c/p^s}] \\ W_n(\mathcal{O}_E/p)/[\mathfrak{a}_E^{\geq c/p^s}] &\simeq W_n(\mathcal{O}_E/\mathfrak{a}_E^{\geq cp^{n-1-s}})/[\mathfrak{a}_E^{\geq c/p^s}]. \end{aligned}$$

Hence $f_{cp^{n-1-s}}$ indeed induces a map

$$W_n(\mathcal{O}_{L_s}/\mathfrak{a}_{L_s}^{\geq cp^{n-1-s}})/[\mathfrak{a}_{L_s}^{\geq c/p^s}] \to W_n(\mathcal{O}_E/\mathfrak{a}_E^{\geq cp^{n-1-s}})/[\mathfrak{a}_E^{\geq c/p^s}]$$

and checking injectivity is now straitforward using (1). □

Thus by Lemma 4.2.3, we get injections:

$$\rho_{a,b}^{(s),L_s}(J_{n,a}^{(s),L_s}(\mathfrak{M})) \hookrightarrow \rho_{a,b}^{(s),E}(J_{n,a}^{(s),E}(\mathfrak{M})) \hookrightarrow \rho_{a,b}^{(s)}(J_{n,a}^{(s)}(\mathfrak{M})) = \rho_{a,b}(J_{n,a}(\mathfrak{M})) \simeq T$$

the first one being induced by f (which is obviously compatible with Frobenius since it is a ring homomorphism). By Theorem 4.1.1, LHS is isomorphic to T. The composite map is then an injective endomorphism of T. Consequently, it is an isomorphism because T is finite. It follows that $\rho_{a,b}^{(s),E}(J_{n,a}^{(s),E}(\mathfrak{M})) \hookrightarrow \rho_{a,b}^{(s)}(J_{n,a}^{(s)}(\mathfrak{M}))$ is bijective and then, applying again Theorem 4.1.1, we get $L_s \subset E$. Property $(P_m^{L_s/K_s})$ is proved.

By Proposition 4.2.1, one then get $\mu_{L_s/K_s} \leq e_{K_s/K} m = \frac{Np^n}{p-1}$. Taking $N = ern$, one obtain Theorem 1.4. (Recall that ern is not in general the best value one can choose (expect for $n = 1$). See §2.4 for a discussion about this.)

4.3. **Proof of Theorem 1.1.** Consider α and β such that $\frac{N}{e(p-1)} = p^\alpha \beta$ with $\alpha \in \mathbb{N}$ and $\frac{1}{p} < \beta \leq 1$. (If $N = ern$, then α and β are those of Theorem 1.1). From now on, we fix $s = n+\alpha$. One certainly have $s \geq s_0(a) = n + \log_p(\frac{N}{e(p-1)})$ as it was assumed in the beginning of this section.

It is very easy now to bound valuation of $\mathcal{D}_{L/K}$. We just write:

$$\begin{aligned} v_K(\mathcal{D}_{L_s/K}) &= 1 + es - \frac{1}{p^s} + v_K(\mathcal{D}_{L_s/K_s}) < 1 + es - \frac{1}{p^s} + ap^{n-1-s} \\ &= 1 + es - \frac{1}{p^s} + ep^{\alpha+n-s}\beta = 1 + e(n+\alpha+\beta) - \frac{1}{p^{n+\alpha}} \end{aligned}$$

where the inequality $v_K(\mathcal{D}_{L_s/K}) < ap^{n-1-s}$ follows from Corollary 4.2.2 and the fact that $(P^{L_s/K_s}_{ap^{n-1-s}})$ holds as it was seen before. Now, since L is a subextension of L_s, we have $v_K(\mathcal{D}_{L/K}) \leq v_K(\mathcal{D}_{L_s/K})$ and the previous bound works also for $v_K(\mathcal{D}_{L/K})$. Taking $N = ern$, we get Theorem 1.1.(2).

To bound $\mu_{L/K}$, we first need to extend the definition of φ_{F_1/F_0} and ψ_{F_1/F_0} to arbitraty finite extensions F_1/F_0 (non necessarily Galois). There are several standard ways to do this. For example, following [26], §1.2.1, one can put

$$\psi_{F_1/F_0}(\mu) = \int_0^\mu [G^{(1)}_{F_0} : G^{(1)}_{F_1} G^{(t)}_{F_0}]\, dt$$

(where G_{F_1} and G_{F_0} stands for absolute Galois groups of F_1 and F_0 respectively) and remark that this formula agrees with the previous definition when F_1/F_0 is Galois. Set also $\varphi_{F_1/F_0} = (\psi_{F_1/F_0})^{-1}$. We have an usual transitivity formula: if $F_0 \subset F_1 \subset F_2$ are finite extensions of K, then $\varphi_{F_2/F_0} = \varphi_{F_1/F_0} \circ \varphi_{F_2/F_1}$.

Lemma 4.3.1. *Let $F_0 \subset F_1$ be two finite extension of K. Then*

$$G^{(\mu)}_{F_1} = G_{F_1} \cap G^{(\varphi_{F_1/F_0}(\mu))}_{F_0}$$

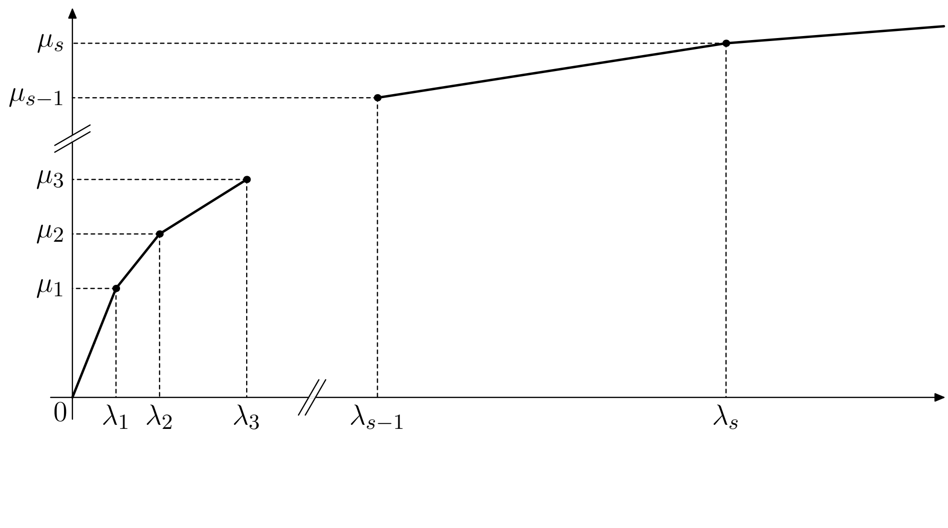
for all $\mu \geq 0$.

Proof. Let N be a normal extension of F_0, with $F_1 \subset N$. For all $\lambda \geq 0$, one have $\mathrm{Gal}(N/F_1)_{(\lambda)} = \mathrm{Gal}(N/F_1) \cap \mathrm{Gal}(N/F_0)_{(\lambda)}$, that is $\mathrm{Gal}(N/F_1)^{(\varphi_{N/F_1}(\lambda))} = \mathrm{Gal}(N/F_1) \cap \mathrm{Gal}(N/F_0)^{(\varphi_{N/F_0}(\lambda))}$. Putting $\mu = \varphi_{N/F_1}(\lambda)$ and using transitivity formula, one get $\mathrm{Gal}(N/F_1)^{(\mu)} = \mathrm{Gal}(N/F_1) \cap \mathrm{Gal}(N/F_0)^{(\varphi_{F_1/F_0}(\mu))}$. Taking projective limit over all Galois extensions N, we get the desired property. $\square$

By §4.2, we know that $G^{(\mu)}_s \subset G_L$ for all $\mu > \frac{Np^n}{p-1}$. Applying previous Lemma, we have $G_s \cap G^{(\varphi_{K_s/K}(\mu))}_K$ also lies in G_L. Consequently

$$\mu_{L/K} \leq \mu_{L_s/K} \leq \max\left(\mu_{K_s/K}, \varphi_{K_s/K}\left(\frac{Np^n}{p-1}\right)\right). \tag{4.3.1}$$

By Remark 5.5 of [16], we know that $\mu_t := \mu_{K_t/K} = 1 + e(t + \frac{1}{p-1})$ for all $t \geq 1$. Using that subextensions of K_s are exactly the K_t's for $0 \leq t \leq s$, one easily see that $\varphi_{K_s/K}$ has the following shape



where successive slopes are $1, \frac{1}{p}, \frac{1}{p^2}, \ldots, \frac{1}{p^s}$. One can then compute λ_t's ($0 \leq t \leq s$) and one find $\lambda_t = 1 + \frac{ep^t}{p-1}$. Since $\varphi_{K_s/K}$ is concave and its last slope is $\frac{1}{p^s}$, one get

$$\varphi_{K_s/K}(\lambda) \leq \mu_s + \frac{\lambda - \lambda_s}{p^s} = 1 + es - \frac{1}{p^s} + \frac{\lambda}{p^s}.$$

Finally, taking $N = ern$ and using (4.3.1), one obtain Theorem 1.1.(1) (remember $s = n + \alpha$ and $\frac{rn}{p-1} = p^\alpha \beta$).

5. Some results and questions about lifts

In this last section, we discuss how to fit our main theorem into the investigation of the following fundamental question: when a p^n-torsion Galois representation comes from geometry? As the question is very difficult to answer, we would like to first consider the relatively easier one: when a p^n-torsion of (local) Galois representation comes from semistable representations? In the following, we formulate the question precisely, provide some partial results and raise several related questions. More precisely, we would like to investigate when a given torsion representation of G_K can be realized as a quotient of two lattices in a semistable (or even crystalline) representation, eventually with prescribed Hodge-Tate weights. Denote by $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$ (resp. $\mathrm{Rep}_{\mathrm{tor}}(G_K)$, resp. $\mathrm{Rep}_{\mathbb{Z}/p^n\mathbb{Z}}(G_K)$) the category of all $\mathbb{Z}_p$-representations of G_K that are finitely generated and free (resp. annihilated by a power of p, resp. annihilated by p^n) as a $\mathbb{Z}_p$-module. For any full subcategory $\mathcal{C}$ of $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$, one can always raise the following question

Question 5.1. For any $T \in \mathrm{Rep}_{\mathrm{tor}}(G_K)$ (resp. $T \in \mathrm{Rep}_{\mathbb{Z}/p^n\mathbb{Z}}(G_K)$), does there exist Λ and Λ' in $\mathcal{C}$ such that $T \simeq \Lambda/\Lambda'$?

Obviously if $\mathcal{C}$ is stable under subobject (which will in general be true in interesting examples), it is enough to find L together with a surjective G_K-equivariant morphism $\Lambda \to T$. In the sequel, we will call a *lift* such a morphism $\Lambda \to T$. If $\mathcal{C}$ is moreover stable by direct sum, the problem can be further reduced as follows.

Proposition 5.2. *Assume that $\mathcal{C}$ is stable under subobjects and direct sums. Assume also that any $T \in \mathrm{Rep}_{\mathbb{Z}/p\mathbb{Z}}(G_K)$ admits a lift $\Lambda \in \mathcal{C}$. Then the answer to Question 5.1 is "yes".*

Proof. We make an induction on n. The case $n = 1$ is obvious. Now assume the statement is valid for $m \leq n - 1$. Let T be a representation annihilated by p^n. Then we have an exact sequence $0 \to T' \to T \to T'' \to 0$, where $T' = p^{n-1}T$ and $T'' = T/T'$. Since T'' is annihilated by p^{n-1}, by induction, there exists an $\Lambda \in \mathcal{C}$ that lifts T''. Denote the surjections $\Lambda \to T''$ and $T \to T''$ by f and g respectively. Set $M := T \times_{T''} \Lambda = \{(x, y) \in T \times \Lambda \,/\, g(x) = f(y)\}$. Then we have an exact sequence $0 \to T' \to M \to \Lambda \to 0$. Since Λ is free over $\mathbb{Z}_p$, the sequence is split as $\mathbb{Z}_p$-module. In particular $pM \simeq p\Lambda \oplus pT' = p\Lambda$ is finite free over $\mathbb{Z}_p$. Now we have exact sequence $0 \to pM \to M \to M' \to 0$ with $M' = M/pM$. Since M/pM is annihilated by p, there exists an $\Lambda' \in \mathcal{C}$ such that Λ' lifts M/pM. Set $N := M \times_{M'} \Lambda'$. It sits in the exact sequence $0 \to pM \to N \to \Lambda' \to 0$, and since pM and Λ' are both finite free, N is also. Note that N is a lift of M hence a lift of T. Now it remains to show that N is in $\mathcal{C}$. To see this, note that $N := M \times_{M'} \Lambda' \subset M \times \Lambda'$. Then $pN \subset (pM) \times p\Lambda'$. But $pM \simeq p\Lambda \in \mathcal{C}$. Hence $pN \subset p\Lambda \times p\Lambda'$ belongs to $\mathcal{C}$. $\square$

We also have a kind of descent property:

Proposition 5.3. *Assume that the answer of question 5.1 is "yes". for the category $\mathcal{C} = \mathcal{C}_K$.*

Let L/K be a finite extension. Denote by $\mathcal{C}_L$ the category whose objects are subrepresentations of restrictions to G_L of objects in $\mathcal{C}_K$. Then, for any $T \in \mathrm{Rep}_{\mathrm{tor}}(G_L)$ (resp. $T \in \mathrm{Rep}_{\mathbb{Z}/p^n\mathbb{Z}}(G_L)$), there exist Λ and Λ' in $\mathcal{C}_L$ such that $T \simeq \Lambda/\Lambda'$.

Proof. By a previous remark, it is enough to show that T admits a lift in $\mathcal{C}$. Let $T_0 := \mathrm{Ind}_{G_K}^{G_L}(T)$. By assumption, there exists a lift $f : \Lambda_0 \to T_0$ with $\Lambda_0 \in \mathcal{C}_K$. Consider the $\mathbb{Z}_p$-linear map $\mathrm{pr} : \mathbb{Z}_p[G_K] \to \mathbb{Z}_p[G_L]$ sending $g \in G_L$ to itself, and $g \in G_K$, $g \notin G_L$ to 0; it is surjective and G_L-equivariant for actions on both sides. Tensoring pr by T, one get a G_L-equivariant surjective morphism $T_0 \to T$ which, composed with f, gives the desired lift. $\square$

Nevertheless, of course, the answer to Question 5.1 is in general negative. For instance, we have the following theorem that can be seen as a consequence of ramification bounds obtained in this paper.

Theorem 5.4. *For any $r > 0$, answer to Question 5.1 is "no" if $\mathcal{C}$ is the category of lattices in semistable representations with Hodge-Tate weights in $\{0, \dots, r\}$.*

Proof. There are several ways to prove this theorem. Below, we give two different methods.

The first one is based on results shown in this paper. Select a Galois extension F/K which has very large ramification and let T be the regular representation with $\mathbb{Z}/p^n\mathbb{Z}$-coefficients of $\mathrm{Gal}(F/K)$. Then the splitting field of T is F, and Theorem 1.1 shows that T cannot in general be lifted a semistable representation with Hodge-Tate weights in $\{0, \dots, r\}$.

The second proof we would like to give uses the main result of [20] which states that a finite free $\mathbb{Z}_p$-representation Λ of G_K is a lattice in a crystalline (resp. semistable) representation with Hodge-Tate weights $\{0, \dots, r\}$ if and only if $\Lambda/p^n\Lambda$ is a quotient of two such lattices for any n. Therefore, starting from a representation Λ such that $\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is not semistable, there must exist an integer n such that $\Lambda/p^n\Lambda$ gives a counter-example to Question 5.1 (with the category $\mathcal{C}$ of the theorem). $\square$

Unfortunately, the above proof does not help us to solve the following more interesting question:

Question 5.5. Has Question 5.1 a positive answer when $\mathcal{C}$ is the category of all lattices in semistable representations?

In fact, to check the above question, it suffices to look at representations annihilated by p (by Proposition 5.2) and we may assume that $K = \mathbb{Q}_p$ (by Proposition 5.3). Here are some partial results in favor of a positive answer to Question 5.5.

Proposition 5.6. *Let T be a torsion representation of G_∞. Then T is a quotient of two representations arising from finite free Kisin modules.*

Proof. By a similar argument as in proof of Proposition 5.2, we may assume that T is annihilated by p. Let M be the étale φ-module over $k((u))$ attached to T (see for instance [11], A. 3). Since any torsion Kisin module can be written as a quotient of two free Kisin modules, it is enough to show that M admits a submodule $\mathfrak{M}$ which is a Kisin module of height r. Let $(e_1, \dots, e_d)$ be a basis of M and A be the matrix with coefficients in $k((u))$ such that

$$(\varphi(e_1), \dots, \varphi(e_d)) = (e_1, \dots, e_d)A.$$

Since changing all e_i's in ue_i changes A in $u^{p-1}A$, one may assume that A has coefficients in $k[[u]]$. Furthermore, the étaleness of Frobenius on M exactly means that A is invertible in $k((u))$. Hence $\det A$ does not vanish. Finally, we choose r such that $\det A$ divides u^{er} (that is $r \geq \frac{1}{e}\mathrm{val}_u(\det A)$) and we are done. $\square$

Theorem 5.7. *Any tamely ramified $\mathbb{F}_p$-representation of G_K can be written as a quotient of two lattices in a crystalline representation with Hodge-Tate weights between 0 and $1 + E(\frac{p-1}{e})$.*

Remark 5.8. In particular, the answer to Question 5.5 is yes if T is tamely ramified and annihilated by p.

Proof. In a preliminary version of this paper, the authors gave a proof based on some computations in p-adic Hodge theory, making in particular an intensive use of results of [19]. The following simplier argument is due to an anonymous referee.

Put $r = 1 + E(\frac{p-1}{e})$ and denote by I the inertia subgroup of G_K. Let T be a tamely ramified representation of G_K annihilated by p. Since the tame inertia group is procyclic of order prime to p, $T|_I$ splits as a direct sum of irreducible representations. By [24], §1.7, every irreducible representation of I is isomorphic to

$$\mathbb{F}_{p^d}\left(\theta_0^{n_0}\theta_1^{n_1}\cdots\theta_{d-1}^{n_{d-1}}\right)$$

where θ_i's are fundamental inertia character of level d (see *loc. cit.*) and n_i's are some integers in $\{0, \dots, p-1\}$. Hence, $T|_I$ can be written as a tensor product of at most r irreducible representations

T_i of I whose tame inertia weights are between 0 and e. By a classical result of Raynaud (see [23]), all T_i come from finite flat group schemes. Using the fact that any finite flat group scheme can be embedded in a p-divisible group, one construct a crystalline lift of T_i with Hodge-Tate weights in $\{0, 1\}$. Taking the tensor product of all these lifts, one get a crystalline representation Λ with Hodge-Tate weights between 0 and r together with a surjective I-equivariant morphism $f : \Lambda \to T$ (which certainly factors through L/pL). Since $\Lambda/p\Lambda$ and T are finite dimensional over $\mathbb{F}_p$, they are finite and f is $G_{K'}$-equivariant for a finite Galois unramified extension K' of K. Consider the map

$$\mathrm{Ind}_{G_{K'}}^{G_K} \Lambda = \mathbb{Z}_p[G_K] \otimes_{\mathbb{Z}_p[G_{K'}]} \Lambda \to T, \quad [\sigma] \otimes x \mapsto \sigma f(x).$$

It is apparently G_K-equivariant and surjective: it is a lift of T. Furthermore, the restriction of $\mathrm{Ind}_{G_{K'}}^{G_K} \Lambda$ to $G_{K'}$ is a direct sum of copies of Λ, and hence is crystalline with Hodge-Tate weights in $\{0, \dots, r\}$. Since K'/K is unramified, also is $\mathrm{Ind}_{G_{K'}}^{G_K} \Lambda$ and we are done. $\square$

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