An introduction to p-adic period rings

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Abstract

This paper is the augmented notes of a course I gave jointly with Laurent Berger in Rennes in 2014. Its aim was to introduce the periods rings B_{crys} and B_{dR} and state several comparison theorems between étale and crystalline or de Rham cohomologies for p-adic varieties.

Contents

Introduction

In algebraic geometry, the word *period* often refers to a complex number that can be expressed as an integral of an algebraic function over an algebraic domain. One of the simplest periods is $2i\pi = \int_\gamma \frac{dt}{t}$, where γ is the unit circle in the complex plane. Equivalently, a period can be seen as an entry of the matrix (in rational bases) of the de Rham isomorphism:

$$\mathbb{C} \otimes_{\mathbb{Q}} H^r_{\text{sing}}(X(\mathbb{C}), \mathbb{Q}) \simeq \mathbb{C} \otimes_K H^r_{\text{dR}}(X) \tag{1}$$

for an algebraic variety X defined over a number field K. (Here H^r_{sing} is the singular cohomology and H^r_{dR} denotes the *algebraic* de Rham cohomology.)

The initial motivation of p-adic Hodge theory is the will to design a relevant p-adic analogue of the notion of periods. To this end, our first need is to find a suitable p-adic generalization of the isomorphism (1). In the p-adic setting, the singular cohomology is no longer relevant; it has to be replaced by the étale cohomology. Thus, what we need is a ring B allowing for a canonical isomorphism:

$$B \otimes_{\mathbb{Q}_p} H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p) \simeq B \otimes_K H^r_{\text{dR}}(X) \tag{2}$$

when K is now a finite extension of $\mathbb{Q}_p$ and X is a variety defined over K. Of course, the first natural candidate one thinks at is $B = \mathbb{C}_p$, the p-adic completion of an algebraic closure $\bar{K}$ of K. Unfortunately, this first period ring does not totally fill our requirements. More precisely, it turns out that $\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$ is isomorphic to the graded module (for the de Rham filtration) of $\mathbb{C}_p \otimes_K H^r_{\text{dR}}(X)$ but not to $\mathbb{C}_p \otimes_K H^r_{\text{dR}}(X)$ itself. The main objective of this lecture is to detail the construction of two periods rings, namely B_{crys} and B_{dR}, allowing for the isomorphism (2) under some additional assumptions on the variety X. The ring B_{dR} (which is the bigger one) is often called the *ring of p-adic periods*.

Another important aspect of p-adic period rings concerns the Galois structure of $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$. Indeed, we shall see that the mere existence of the isomorphism (2) usually has strong consequences on the Galois module $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$. In order to give depth to this observation, Fontaine developed a general formalism for studying and classifying general Galois representations through the notion of period rings. A large part of this article focuses on the Galois aspects.

Structure of the article. §1 serves as a second long introduction to this article; two results which can be considered as the seeds of p-adic Hodge theory are presented and discussed. The first one is due to Tate and provides a Hodge-like decomposition of the Tate module of a p-divisible group in the spirit of the isomorphism (2). The second result is a classification theorem of p-divisible groups by Fontaine. Fontaine's general formalism for studying Galois representations is also introduced in this section.

In §2, we investigate to what extent $\mathbb{C}_p$ meets the expected properties of a period ring. We adopt the point of view of Galois representations, which means concretely that we will concentrate on isolating those Galois representations that are susceptible to sit in an isomorphism of the form (2) when $B = \mathbb{C}_p$. This study will lead eventually to the notion of Hodge–Tate representations, which is related to the Hodge-like decompositions of cohomology presented in §1.

In §3, we review the construction of the period rings B_{crys} and B_{dR}; it is the heart of the article but also its most technical part. Finally, in §4, we state several comparison theorems between étale and de Rham cohomologies. We also show how the rings B_{crys} and B_{dR} intervene in the classification of Galois representations, through the notions of crystalline and de Rham representations.

Some advice to the reader. Although we will give frequently reminders, we assume that the reader is familiar with the general theory of local fields as presented in [39], Chapter 1–4. A minimal knowledge of local class field theory [39] and of the theory of p-adic analytic functions [33] is also welcome, while not rigourously needed.

To the impatient reader who is afraid by the length of this article and is not interested in the details of the proofs (at least in first reading) but only by a general outline of p-adic Hodge theory, we advise to read §1, then the introduction of §3 until §3.1 and then finally §4.

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Notations. Throughout this article, the letter p will refer to a fixed prime number. We use the notation $\mathbb{Z}_p$ (resp. $\mathbb{Q}_p$) for the ring of p-adic integers (resp. the field of p-adic numbers). We recall that $\mathbb{Q}_p = \text{Frac}\, \mathbb{Z}_p = \mathbb{Z}_p[\frac{1}{p}]$. Let also $\mathbb{F}_p$ denote the finite field with p elements, *i.e.* $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

If $\mathfrak{A}$ in a ring, we denote by $\mathfrak{A}^\times$ the multiplicative group of invertible elements in $\mathfrak{A}$.

1 From Hodge decomposition to Galois representations

After having recalled some basic facts about local fields in §1.1, we discuss in §1.2 two families of results which are the seeds of p-adic Hodge theory. Both of them are of geometric nature. The first one concerns the classification of p-divisible groups over the ring of integers of a local field, while the second one concerns the Hodge-like decomposition of the étale cohomology of varieties defined over local fields. From this presentation, the need to have a good tannakian formalism emerges.

Carried by this idea, we move from geometry to the theory of representations and focus on tensor products and scalar extensions. Eventually, this will lead us to the notion of B-admissibility, which is the key concept in Fontaine's vision of p-adic Hodge theory. Finally, we briefly discuss the applications we will develop in the forthcoming sections: using B-admissibility, we introduce the notions of crystalline, semi-stable and de Rham representations and explain rapidly how the general theory can help for studying these classes of representations.

1.1 Setting and preliminaries

Let K be a finite extension[1] of $\mathbb{Q}_p$. Let $v_p : K \to \mathbb{Q} \sqcup \{+\infty\}$ be the valuation on K normalized by $v_p(p) = 1$. By our assumptions, $v_p(K^\times)$ is a discrete subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$; hence it is equal to $\frac{1}{e}\mathbb{Z}$ for some positive integer e. We recall that this integer e is called the *absolute ramification index* of K. A *uniformizer* of K is an element of minimal positive valuation, that is of valuation $\frac{1}{e}$. We fix a uniformizer π of K.

Let $\mathcal{O}_K$ be the ring of integers of K, that is the subring of K consisting of elements with nonnegative valuation. We recall that $\mathcal{O}_K$ is a local ring whose maximal ideal $\mathfrak{m}_K$ consists of elements with positive valuation. The residue field k of K is, by definition, the quotient $\mathcal{O}_K/\mathfrak{m}_K$. Under our assumptions, k is a finite field of characteristic p.

Let $W(k)$ denote the ring of Witt vectors with coefficients in k. Set $K_0 = \text{Frac}\, W(k)$. By the general theory of Witt vectors, there exists a canonical embedding $K_0 \to K$. Moreover, through this embedding, K appears as a finite totally ramified extension of K_0 of degree e. Therefore, K_0 is the maximal subextension of K which is unramified over $\mathbb{Q}_p$.

1.1.1 The absolute Galois group of K

We choose and fix once for all an algebraic closure $\bar{K}$ of K. We recall that the valuation v_p extends uniquely to $\bar{K}$, so that we can talk about the ring of integers $\mathcal{O}_{\bar{K}}$ of $\bar{K}$. This ring is a local ring whose maximal ideal will be denoted by $\mathfrak{m}_{\bar{K}}$. The quotient $\mathcal{O}_{\bar{K}}/\mathfrak{m}_{\bar{K}}$ is identified with an algebraic closure of k; it will be denoted $\bar{k}$ in the sequel.

Let $G_K = \text{Gal}(\bar{K}/K)$ be the absolute Galois group of K. Any element of G_K acts by isometry on $\bar{K}$ and therefore stabilizes $\mathcal{O}_{\bar{K}}$ and $\mathfrak{m}_{\bar{K}}$. It thus acts on the residue field $\bar{k}$. This defines a group homomorphism $G_K \to \text{Gal}(\bar{k}/k)$, which is surjective. The kernel of this morphism is the *inertia* subgroup; we shall denote it by I_K in the sequel. The subextension of $\bar{K}$ cut out by I_K

[1] We could have considered a more general setting where K is a complete discrete valued field of characteristic 0 with perfect residue field of characteristic p. All the results presented in the paper extend to this more general setting. However the case of finite extensions of $\mathbb{Q}_p$ is the main case of interest and restricting to this case simplifies the exposition at several points.

is the maximal unramified extension of K; we will denote it by K^{ur}. Summarizing the above discussion, we find that G_K sits in the following exact sequence:

$$1 \longrightarrow I_K \longrightarrow G_K \longrightarrow \mathrm{Gal}(\bar{k}/k) \to 1.$$

The structure of $\mathrm{Gal}(\bar{k}/k)$ is also known: if k has cardinality q, $\mathrm{Gal}(\bar{k}/k)$ is the profinite group generated by the Frobenius $\mathrm{Frob}_q : x \mapsto x^q$.

The structure of I_K can be further precised. Indeed a simple application of Hensel's lemma shows that any finite extension of K^{ur} whose degree is not divisible by p has the form $K^{\mathrm{ur}}[\sqrt[n]{\pi}]$. The union of all these extensions is called K^{tr}; it is the *maximal tamely ramified* extension of K. Since K^{ur} contains all n-th roots of unity for $n \nmid p$ (cf the paragraph *The cyclotomic extension* below for more details), the extension $K^{\mathrm{tr}}/K^{\mathrm{ur}}$ is Galois and its Galois group is identified with $\varprojlim_{n, p\nmid n} \mathbb{Z}/n\mathbb{Z} \simeq \prod_{\ell \neq p} \mathbb{Z}_\ell$. Moreover, any finite extension of K^{tr} has degree p^m for some integer m. On the Galois side, these properties imply that the closed subgroup of I_K corresponding to the extension K^{tr} is the unique pro-p-Sylow of I_K (which is then a normal subgroup) and that I_K sits in the following exact sequence:

$$1 \longrightarrow P_K \longrightarrow I_K \longrightarrow \varprojlim_{n, p\nmid n} \mathbb{Z}/n\mathbb{Z} \to 1$$

where P_K denotes the pro-p-Sylow of I_K.

1.1.2 The cyclotomic extension

The cyclotomic extension of K plays a quite important role in p-adic Hodge theory. So we take some time to recall its most important properties. Let $\mu_n \in \bar{K}$ be a primitive n-th root of unity. We recall that, by definition, the cyclotomic extension of K is the subextension K_{cycl} of $\bar{K}$ generated by the μ_n's.

The extension $K(\mu_n)/K$ is Galois and its Galois group canonically embeds into $(\mathbb{Z}/n\mathbb{Z})^\times$ through the map $\chi_n : \mathrm{Gal}(K(\mu_n)/K) \to (\mathbb{Z}/n\mathbb{Z})^\times$ defined by the relation $\chi_n(\mu_n) = \mu_n^{\chi_n(g)}$ for all $g \in \mathrm{Gal}(K(\mu_n)/K)$. We draw the reader's attention to the fact that χ_n is in general not surjective although it is for all n when $K = \mathbb{Q}_p$.

When n is coprime with p, the extension $K(\mu_n)/K$ is unramified since the polynomial $X^n - 1$ splits over $\bar{k}$. In this case, $K(\mu_n)$ appears as a subextension of K^{ur}. On the other hand, when $n = p^r$ is a power of p, the extension $K(\mu_{p^r})/K$ is totally ramified. This dichotomy motivates the introduction of the two following infinite extensions of K:

$$K_{p'\text{-cycl}} = \bigcup_{n, p\nmid n} K(\mu_n) \quad \text{and} \quad K_{p\text{-cycl}} = \bigcup_{r \geq 0} K(\mu_{p^r}).$$

The first one is actually equal to K^{ur} since, at the level of residue fields, $\bar{k}$ is obtained by k by adding all p^n-th roots of unity for $p \nmid n$. As for $K_{p\text{-cycl}}$, it is linearly disjoint from K^{ur}. It is sometimes called the *p-cyclotomic* extension of K. Clearly, the cyclotomic extension of K is the compositum of K_{ur} and $K_{p\text{-cycl}}$.

Let us review briefly the Galois properties of $K_{p\text{-cycl}}$. First of all, we notice that $K_{p\text{-cycl}}/K$ is Galois. Its Galois group is equipped with an injective group homomorphism $\chi_{p^\infty} : \mathrm{Gal}(K_{p\text{-cycl}}/K) \to \mathbb{Z}_p^\times$ which is characterized by the relation $g\mu_{p^m} = \mu_{p^m}^{\chi_{p^\infty}(g)}$ (for all $g \in \mathrm{Gal}(K_{p\text{-cycl}}/K)$ and $m \geq 1$). Let $\chi_{\mathrm{cycl}} : G_K \to \mathbb{Z}_p^\times$ be the homomorphism obtained by precomposing χ_{p^∞} with the canonical surjection $G_K \to \mathrm{Gal}(K_{p\text{-cycl}}/K)$. We shall often see χ_{cycl} as a character and will call it the *(p-adic) cyclotomic character*. As χ_{p^∞}, it is determined by the relation:

$$g\mu_{p^m} = \mu_{p^m}^{\chi_{\mathrm{cycl}}(g)} \quad \text{for all } g \in G_K \text{ and } m \geq 1.$$

By construction, the extension corresponding to $\ker \chi_{\text{cycl}}$ is $K_{p\text{-cycl}}$ and, more generally, for all positive integer r, the extension corresponding to $\ker(\chi_{\text{cycl}} \bmod p^r)$ is $K(\mu_{p^r})$.

The logarithm defines a group morphism $\mathbb{Z}_p^\times \to \mathbb{Z}_p$ where the group structure on the target is given by the addition. It sits in the exact sequence:

$$1 \longrightarrow \mathbb{F}_p^\times \xrightarrow{[\cdot]} \mathbb{Z}_p^\times \xrightarrow{\log} \mathbb{Z}_p \longrightarrow 1 \qquad (3)$$

where $[\cdot]$ denotes the Teichmuller representative function. This sequence is split since a retraction of $\mathbb{F}_p^\times \to \mathbb{Z}_p^\times$ is simply the canonical projection. Therefore $\mathbb{Z}_p^\times$ is canonically isomorphic to $\mathbb{F}_p^\times \times \mathbb{Z}_p$. Restricting (3) to the image of χ_{cycl}, we find that $\text{Gal}(K_{p\text{-cycl}}/K)$ sits in another exact sequence which reads as follows:

$$1 \longrightarrow H \longrightarrow \text{Gal}(K_{p\text{-cycl}}/K) \xrightarrow{\log \chi_{\text{cycl}}} p^{r_0}\mathbb{Z}_p \longrightarrow 1.$$

Here r_0 is a nonnegative integer and H can be identified as a subgroup of $\mathbb{F}_p^\times$ and thus is cyclic of order divisible by $p-1$. The above sequence splits, so that $\text{Gal}(K_{p\text{-cycl}}/K)$ is canonically isomorphic to a direct product $H \times p^{r_0}\mathbb{Z}_p \simeq H \times \mathbb{Z}_p$.

The subextension of $K_{p\text{-cycl}}$ cut out by the factor $\mathbb{Z}_p$ is nothing but $K(\mu_p)$. It is also the maximal tamely ramified subextension of $K_{p\text{-cycl}}$. The Galois group of $K_{p\text{-cycl}}/K(\mu_p)$ is canonically isomorphic to $\mathbb{Z}_p$ *via* the additive character $p^{-r_0} \log \chi_{\text{cycl}}$. We say that $K_{p\text{-cycl}}/K(\mu_p)$ is a $\mathbb{Z}_p$-extension. The fact that $\text{Gal}(K_{p\text{-cycl}}/K)$ splits as a direct product means that this extension descends to K; in particular, K itself admits a $\mathbb{Z}_p$-extension.

1.1.3 Characters of $G_{\mathbb{Q}_p}$

The representation theory of G_K is the main object of interest in this article. Among all representations of G_K, the simplest ones are of course characters, which are representations of dimension 1. We have actually already seen an example of such character: the cyclotomic character χ_{cycl}. From χ_{cycl}, we can build the following other character:

$$\omega_{\text{cycl}} : G_K \xrightarrow{\chi_{\text{cycl}}} \mathbb{Z}_p^\times \xrightarrow{\bmod p} \mathbb{F}_p^\times \xrightarrow{[\cdot]} \mathbb{Z}_p^\times$$

where the last map takes an element to its Teichmüller representative. We observe that ω_{cycl} is a finite order character, whose order divides $p-1$. When $K = \mathbb{Q}_p$, the order of ω_{cycl} is exactly $p-1$.

Another quite important family of characters are unramified characters, that are those characters which are trivial on the inertia subgroup. Since $G_K/I_K \simeq \text{Gal}(\bar{k}/k)$ is procyclic, continuous unramified characters are easy to describe: they are all of the form

$$\mu_\lambda : G_K \longrightarrow G_K/I_K \simeq \text{Gal}(\bar{k}/k) \xrightarrow{\text{Frob}_q \mapsto \lambda} \mathbb{Z}_p^\times$$

for λ varying in $\mathbb{Z}_p^\times$.

Using local class field theory (*cf* [39]), it is possible to describe explicitely all characters of G_K. Indeed such characters all factor through the abelianization of G_K, which is closely related to $K^\times$ through the Artin reciprocity map. When $K = \mathbb{Q}_p$, this answer is given by the following proposition.

Proposition 1.1.1. *We assume $p > 2$. Let χ be a character of G_K with values in $\mathbb{Q}_p^\times$. Then, there exist unique $\lambda \in \mathbb{Z}_p^\times$, $a \in \mathbb{Z}_p$ and $b \in \mathbb{Z}/(p-1)\mathbb{Z}$ such that $\chi = \mu_\lambda \cdot \chi_{\text{cycl}}^a \cdot \omega_{\text{cycl}}^b$.*

Proof. We first observe that, by compacity, χ must take its values in $\mathbb{Z}_p^\times$. By the Kronecker–Weber theorem, we know that the maximal abelian extension of $\mathbb{Q}_p$ is the cyclotomic extension. Therefore χ has to factor through $\text{Gal}(\mathbb{Q}_{p,\text{cycl}}/\mathbb{Q}_p)$. In particular, $\chi_{|I_{\mathbb{Q}_p}}$ factors through $\text{Gal}(\mathbb{Q}_{p,\text{cycl}}/\mathbb{Q}_p^{\text{ur}})$ which is isomorphic to $\mathbb{Z}_p^\times$ by the cyclotomic character. Consequently, $\chi_{|I_{\mathbb{Q}_p}} = h \circ \chi_{\text{cycl}}$ for some group homomorphism $h : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$. Moreover, when $p > 2$, we have an isomorphism $\mathbb{Z}_p^\times \simeq \mathbb{F}_p^\times \times \mathbb{Z}_p$, $x \mapsto (x \bmod p, \log x)$, the inverse being given by $(a, b) \mapsto [a] \cdot \exp b$ where $[a]$ denotes the Teichmuller representative of a. From this description, we derive that there exist $a \in \mathbb{Z}_p$ and $b \in \mathbb{Z}/(p-1)\mathbb{Z}$ such that $h(x) = x^a \cdot [x \bmod p]^b$ for $x \in \mathbb{Z}_p^\times$. Thus $\chi_{I_{\mathbb{Q}_p}} = \chi_{\text{cycl}}^a \cdot \omega_{\text{cycl}}^b$. The character $\chi \cdot \chi_{\text{cycl}}^{-a} \cdot \omega_{\text{cycl}}^{-b}$ is then unramified. Thus it must be of the form μ_λ for some $\lambda \in \mathbb{Z}_p^\times$. The proposition is proved. □

1.2 Motivations: p-divisible groups and étale cohomology

The starting point of p-adic Hodge theory is Tate's paper of 1966 on p-divisible groups [40]. In this seminal article, Tate establishes a Hodge-like decomposition of the Tate module of a p-divisible group on $\mathcal{O}_K$. More precisely, let $\mathcal{G}$ be a p-divisible group on $\mathcal{O}_K$. We define the Tate module of $\mathcal{G}$ by $T_p\mathcal{G} = \varprojlim_n \mathcal{G}[p^n](\bar{K})$. Observe that $T_p\mathcal{G}$ is naturally endowed with an action of G_K. The algebraic structure of $T_p\mathcal{G}$ is well-known: it is a free module of finite rank over $\mathbb{Z}_p$. Set $V_p\mathcal{G} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p\mathcal{G}$. Then, Tate proves the following Hodge-like G_K-equivariant decomposition:

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} V_p\mathcal{G} \simeq \big(\mathbb{C}_p \otimes_{\mathcal{O}_K} \omega_{\mathcal{G}^\vee}\big) \oplus \big(\mathbb{C}_p(\chi_{\text{cycl}}^{-1}) \otimes_{\mathcal{O}_K} \omega_{\mathcal{G}}^\vee\big). \tag{4}$$

Here $\mathcal{G}^\vee$ is the Cartier dual of $\mathcal{G}$, the construction $\omega_{\_}$ refers to the cotangent space at the origin and $\mathbb{C}_p(\chi_{\text{cycl}}^{-1})$ is $\mathbb{C}_p e$ endowed with the action $g(\lambda e) = g\lambda \cdot \chi_{\text{cycl}}^{-1}(g) \cdot e$ (for $g \in G_K$ and $\lambda \in \mathbb{C}_p$). Note that the Galois action is trivial over $\omega_{\mathcal{G}^\vee}$ and $\omega_{\mathcal{G}}^\vee$. The isomorphism (4) then reveals the Galois action on the Tate module. Tate's theorem implies in particular that, when A is an abelian variety over K with good reduction, the étale cohomology of A admits the following decomposition:

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^1_{\text{ét}}(A_{\bar{K}}, \mathbb{Q}_p) \simeq \big(\mathbb{C}_p \otimes_K H^1(A, \mathcal{O}_A)\big) \oplus \big(\mathbb{C}_p(\chi_{\text{cycl}}^{-1}) \otimes_K H^0(A, \Omega_{A/K})\big). \tag{5}$$

were $A_{\bar{K}} = \text{Spec } \bar{K} \times_{\text{Spec } K} A$, $\mathcal{O}_A$ is the structural sheaf of A and $\Omega_{A/K}$ is the sheaf of Kähler differentials of A over K. We refer to Freixas' lecture in this volume [25] for a more detailed discussion—including a sketch of the proof—about Tate's theorem.

After Tate's results, p-divisible groups over various bases were studied intensively. In the 1970's, Fontaine [18] obtained a complete classification of p-divisible groups and finite flat group schemes over $\mathcal{O}_K$ when $K/\mathbb{Q}_p$ is unramified. The starting point of Fontaine's theorem is the classification of p-divisible groups over perfect fields of characteristic p in terms of Dieudonné modules [13]. Let us recall briefly how it works. If $\mathcal{G}_k$ is a p-divisible group over k, we define

$$M(\mathcal{G}_k) = \text{Hom}_{\text{gr}}(\mathcal{G}_k, CW_k) \tag{6}$$

where CW_k is the functor of *Witt covectors* and the notations Hom_{gr} means that we are considering the set of all natural transformations preserving the group structure. The space $M(\mathcal{G}_k)$ is a Dieudonné module. This means that it is a module over $W(k)$ endowed with a Frobenius F (which is a semi-linear endomorphism with respect to the Frobenius on $W(k)$) and a Verschiebung V (which is a semi-linear endormorphism with respect to the inverse of the Frobenius) with the property that $FV = VF = p$. One can show that M realizes an anti-equivalence of categories between the category of p-divisible groups over k and that of finite free Dieudonné modules over $W(k)$, the inverse functor being given by the formula

$$\mathcal{G}_k(A) = \text{Hom}_{W(k),F,V}\big(M, CW(A)\big) \quad \text{for any } k\text{-algebra } A$$

which is quite similar to (6). Now, if $\mathcal{G}$ is a p-divisible over $\mathcal{O}_K$ with special fibre $\mathcal{G}_k$, Fontaine constructs a submodule $L(\mathcal{G}) \subset M(\mathcal{G}_k)$ and demonstrates that it obeys to a certain list of properties. Taking these properties as axioms, Fontaine introduces the notion of *Honda systems* and proves that the association $\mathcal{G} \mapsto (M(\mathcal{G}_k), L(\mathcal{G}))$ is an anti-equivalence of categories between the category of p-divisible groups over $\mathcal{O}_K$ and the category of finite free Honda systems over

$W(k)$. Moreover, Fontaine establishes a compact formula for the inverse functor. This formula reads:

$$V_p\mathcal{G} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{Hom}_{\mathrm{honda}}\big((M(\mathcal{G}_k), L(\mathcal{G})),\ (\mathcal{B}, L_{\mathcal{B}})\big) \tag{7}$$

where the notation $\mathrm{Hom}_{\mathrm{honda}}$ means that we are taking the morphisms in the category of Honda system and the target $(\mathcal{B}, L_{\mathcal{B}})$ is a special Honda system[2] (the letter $\mathcal{B}$ refers to the mathematician Barsotti, who first studied p-divisible groups using this kind of techniques). Moreover, $(\mathcal{B}, L_{\mathcal{B}})$ is endowed with an action of G_K, from which we can recover the G_K-action on $V_p\mathcal{G}$. Compared to Tate's decomposition formula (4), Fontaine's result is more precise because it describes the Tate module $V_p\mathcal{G}$ itself, whereas Tate's result only concerns its scalar extension to $\mathbb{C}_p$. For many complements about Fontaine's classification results, we refer to [18, 12].

About ten years later, in 1981, Fontaine came back to Tate's decomposition isomorphism (5)and gave a different proof of it (which is sketched in Freixas' lecture in this volume [25]), relaxing at the same time the assumption of good reduction. He also became interested in generalizing Tate's decomposition theorem to higher cohomology group (*i.e.* $H^r_{\text{ét}}(A_{\bar{K}}, \mathbb{Q}_p)$ with $r > 1$) and other types of varieties. Moreover, noticing that the right hand side of (5) is the graded module of the de Rham cohomology, one may wonder if one can make the isomorphism (5) more precise and relate the étale cohomology with the de Rham cohomology equipped with its filtration. All these questions had been a strong motivation for the development of p-adic Hodge theory for many years. Nowadays, all of them are solved: it has been proved independently by Faltings [15] and Tsuji [41] that $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p) \simeq B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^r_{\mathrm{dR}}(X)$ whenever X is a proper smooth variety over K and r is a nonnegative integer. Here B_{dR} is the so-called *field of p-adic periods*. We will introduce it in this article in §3. Taking the grading in the above isomorphism, we get the following Hodge-like decomposition:

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p) \simeq \bigoplus_{a+b=r} \mathbb{C}_p(\chi_{\mathrm{cycl}}^{-a}) \otimes_K H^b(X, \Omega^a_{X/K}).$$

We will come back to these results in §4.1.

Finally, it is interesting to confront the two directions of research discussed above, namely classification of p-divisible groups and Hodge-like decomposition theorems. As already mentioned, one important feature of the isomorphism (7) is the fact that it gives a complete description of the Galois action on the Tate module. On the other hand, it is apparent that Honda systems have important limitations: by design, they can only deal with Tate modules, that is, roughly speaking, with the first cohomology group. Analyzing carefully the situation, Fontaine realized that what is missing to Honda systems is a good tannakian formalism (which is, of course, a key point in the line of Hodge-like decomposition theorems). In more crude terms, the fact that we are limited to the $H^1_{\text{ét}}$ should be understood as a reflection of the fact that we are missing a good notion of tensor product on p-divisible groups. As explained in the introduction of [19], the period ring B_{crys} and the afferent notion of crystalline representations actually emerge when trying to conceal the theory of Honda systems with the tannakian formalism inspired from the Hodge-like decomposition theorems we have presented above.

All the developments we will present in the sequel are stamped by this simple idea that one wants to keep apparent the tannakian structure (*i.e.* the tensor product) everywhere and, even, to use it as a main tool. The natural framework in which the theory grows is then that of Galois representations, which has a strong tannakian structure.

1.3 Notion of semi-linear representations

The Hodge-like decomposition theorems discussed previously motivate the study of representations of the form $W = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ where V is a given $\mathbb{Q}_p$-representation of G_K. Since G_K does

[2] Its construction is subtle and we will not give it here. However, we would like to encourage the reader to look at it in Fontaine's paper [18, Chap. V, §1] because it is instructive to realize that it is actually quite close to the construction of the periods B_{dR} and B_{crys} we shall detail in §3.

act on $\mathbb{C}_p$, we observe that W is *not* a $\mathbb{C}_p$-linear representation in the usual sense. Instead, it is a so-called *semi-linear representation*. The aim of this subsection is to introduce and study this notion.

1.3.1 Definitions

In what follows, we let G be a topological group[3] and B be a topological ring equipped with a continuous[4] action of G, which is compatible with the ring structure, *i.e.* $g \cdot (a + b) = ga + gb$ and $g \cdot (ab) = ga\, gb$ for all $g \in G$ and $a, b \in B$.

Definition 1.3.1. A *B-semi-linear representation of G* is the datum of a B-module W equipped with a continuous action of G such that:

$$g \cdot (x + y) = gx + gy \quad \text{and} \quad g \cdot (ax) = ga \cdot gx$$

for all $g \in G$, $a \in B$ and $x, y \in W$.

Clearly, if G acts trivially on B, the notion of B-semi-linear representation of G agrees with the usual notion of B-linear representation of G.

By our assumptions, B itself (endowed with its G-action) is a B-semi-linear representation of G. Similarly we can turn B^n into a B-semi-linear representation by letting G act coordinate by coordinate. The latter representation will be called the *trivial representation* of dimension n.

If W_1 and W_2 are two B-semi-linear representations of G, a morphism $W_1 \to W_2$ is a B-linear mapping which commutes with the action of G. With this definition, we can form the category of B-semi-linear representations of G (for G and B fixed). In the sequel we will simply denote it $\text{Rep}_B(G)$. It is easily seen that $\text{Rep}_B(G)$ is an abelian category. It is moreover endowed with a notion of tensor product and internal hom: if W_1 and W_2 are objects of $\text{Rep}_B(G)$, then $W_1 \otimes_B W_2$ (equipped with the action $g \cdot (x \otimes y) = gx \otimes gy$) and $\text{Hom}_B(W_1, W_2)$ (equipped with the action $g\varphi : x \mapsto g\varphi(g^{-1}x)$) are also.

Scalar extension There is also a natural notion of scalar extension in the framework of semi-linear representations. To explain it, let us consider a closed subring C of B, which is stable under the action of G. Then the notion of C-semi-linear representations of G makes sense and there is a canonical functor $\text{Rep}_C(G) \to \text{Rep}_B(G)$ taking W to $B \otimes_C W$.

The latter construction is quite interesting because it allows us to build semi-linear representations from classical representations. Indeed, assume that we are given a field E and we have chosen G and B is such a way that B is an algebra over E and G acts trivially on E. (As an example, B could be a Galois extension of E with $G = \text{Gal}(B/E)$.) The scalar extension then defines a functor $\text{Rep}_E(G) \to \text{Rep}_B(G)$. Moreover, since the action of G on E is trivial, the category $\text{Rep}_E(G)$ is just the category of E-linear representations of G. In more concrete terms, if V is a classical representation of G defined over E, then $W = B \otimes_E V$ is a B-semi-linear representation. This is actually the prototype of all the semi-linear representations we are going to consider in this article.

Specializing the previous recipe to 1-dimensional representations, we obtain a way to construct semi-linear representations of G from characters of G. Concretely, if $\chi : G \to E^\times$ is a multiplicative character, we will denote by $B(\chi)$ the 1-dimensional representation generated by a vector e_χ on which G acts by $ge_\chi = \chi(g) \cdot e_\chi$ for all $g \in G$. By semi-linearity, we then have

$$g(a\, e_\chi) = ga \cdot \chi(g)\, e_\chi$$

for all $g \in G$ and $a \in B$.

[3]In the application we have in mind, G will be the absolute Galois group of a p-adic field. However, for now, it is better to allow more flexibility and let G be an arbitrary topological group.

[4]By continuous, we mean that the map $G \times B \to B$, $(g, x) \mapsto gx$ is continuous.

1.3.2 Recognizing the trivial representation

We keep the setup of the previous subsection: G is a topological group which acts continuously on a topological ring B.

Definition 1.3.2. A B-semi-linear representation of G is *trivial* if it is isomorphic to the trivial representation B^d for some positive integer d.

While it is in general easy to recognize when a linear representation is trivial (it suffices to check that G acts trivially on each vector of the representation), the task becomes more complicated in the context of semi-linear representations. Indeed, coming back to the definition, we see that a B-semi-linear representation of G is trivial if and only if it admits a basis of vectors which are fixed by G. In particular, it is quite possible that a nontrivial semi-linear representation becomes trivial after scalar extension. The latter remark is in fact the starting point of Fontaine's strategy for classifying Galois representations.

We will discuss Fontaine's strategy in much more details in §1.4. Before this, we have to introduce further notations. Given $W \in \mathrm{Rep}_B(G)$, we denote by W^G the subset of W consisting of fixed points under G, that is the subset of elements $x \in W$ such that $gx = x$ for all $g \in G$. Clearly W^G is a module over B^G. Moreover scalar extension provides a canonical morphism in $\mathrm{Rep}_B(G)$:

$$\alpha_W : B \otimes_{B^G} W^G \longrightarrow W.$$

This morphism is useful for recognizing trivial representations. Indeed it is clearly an isomorphism when W is trivial in the sense of Definition 1.3.2 (since $(B^d)^G = (B^G)^d$) and the converse also holds true when W and W^G are free of finite rank over B and B^G respectively.

1.3.3 Hilbert's theorem 90

As an introduction to Fontaine's strategy, we propose to discuss an easy case where trivial semi-linear representations do appear, while they were not expected at first glance. The setting here is the following. We assume that B is a field and, in order to limit confusion, we will call it L. We assume also that G is a finite group, endowed with the discrete topology. Under these assumptions, L^G is a subfield of L and the extension L/L^G is finite and Galois with Galois group G.

Theorem 1.3.3. *We keep the notations and assumptions above.*
For all $W \in \mathrm{Rep}_L(G)$, the following assertions hold:

1. *the morphism α_W is surjective,*
2. *if W is finite dimensional over L, then α_W is an isomorphism,* i.e. *W is trivial.*

Proof. Let $\lambda_1, \dots, \lambda_n$ be a basis of L over L^G. By Artin's linear independence theorem, there exist constants $\mu_1, \dots, \mu_n \in L$ such that $\sum_{i=1}^n \mu_i g(\lambda_i)$ is 1 if g is the identity and 0 otherwise. Define the trace function $T : W \to W$ by $T(x) = \sum_{g \in G} gx$. One easily checks that T takes its values in W^G. Moreover, for the particular μ_i's we have introduced earlier, we have $\sum_{i=1}^n \mu_i T(\lambda_i x) = x$ for all $x \in W$. This shows the surjectivity of α_W.

We now assume that W is finite dimensional over L. The proof of injectivity is quite similar to the proof of Artin's linear independence theorem. It is enough to check that every finite family of elements of W^G which is linearly independent over L^G remains linearly independent over L. Let then $(x_1, \dots, x_m)$ be a linearly independent family over L^G with $x_i \in W^G$ for all i. We assume by contradiction that there exists a nontrivial relation of linear dependance of the form:

$$a_1 x_1 + a_2 x_2 + \cdots + a_m x_m = 0 \qquad (8)$$

with $a_i \in L$. We choose such a relation so that the number of nonzero a_i's is minimal. Up to reindexing the a_i's and rescaling the relation, we may assume that $a_1 = 1$. Let $g \in G$. Applying $(g - \text{id})$ to (8), we get the relation $(ga_2 - a_2)x_2 + \cdots + (ga_m - a_m)x_m = 0$ which is shorter than (8). From our minimality assumption, we deduce that $ga_i = a_i$ for all $i \geq 2$. Since this is valid for all $g \in G$, we deduce that the linear dependance relation (8) has coefficients in L^G. This is a contradiction since we have assumed that the family $(x_1, \dots, x_m)$ is linearly independent over L^G. □

Remark 1.3.4. Theorem 1.3.3 is often referred to as Hilbert's theorem 90. The reason is that it can be rephrased in the language of group cohomology, then asserting that $H^1(G, \text{GL}_d(L))$ is reduced to one element. This latter statement is an extension of the classical Hilbert's theorem 90 to higher d.

Example 1.3.5. We emphasize that Theorem 1.3.3 does not hold in general when $G = \text{Gal}(L/K)$ where L/K is an *infinite* extension and G is equipped with its natural profinite topology. As an example, take $G = G_{\mathbb{Q}_p} = \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ and let it act on $L = \bar{\mathbb{Q}}_p$. The fixed subfield L^G is $\mathbb{Q}_p$. Consider the semi-linear representation $\mathbb{Q}_p(\chi_{\text{cycl}})$ where we recall that χ_{cycl} denotes the cyclotomic character $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathbb{Z}_p^\times \subset \mathbb{Q}_p^\times$. We claim that $\mathbb{Q}_p(\chi_{\text{cycl}})$ is not isomorphic to $\mathbb{Q}_p$ in the category $\text{Rep}_{\bar{\mathbb{Q}}_p}(G_{\mathbb{Q}_p})$. Indeed, assume by contraction that there exists a G-equivariant isomorphism $\bar{\mathbb{Q}}_p \simeq \bar{\mathbb{Q}}_p(\chi_{\text{cycl}})$. Then there should exist an element $x \in \bar{\mathbb{Q}}_p$ such that

$$gx = \chi(g)\, x \quad \text{for all } g \in G_{\mathbb{Q}_p}. \tag{9}$$

Since x is in $\bar{\mathbb{Q}}_p$, it belongs to a finite extension L of $\mathbb{Q}_p$. Let $N_{L/\mathbb{Q}_p} : L \to \mathbb{Q}_p$ be the norm map from L to $\mathbb{Q}_p$. Applying it to (9), we get the relation $N_{L/\mathbb{Q}_p}(x) = \chi(g)^{[L:\mathbb{Q}_p]} \cdot N_{L/\mathbb{Q}_p}(x)$. Since $N_{L/\mathbb{Q}_p}(x)$ does not vanish, we end up with $\chi(g)^{[L:\mathbb{Q}_p]} = 1$ for all $g \in G_{\mathbb{Q}_p}$, which is a contradiction.

Remark 1.3.6. Similarly, we shall see later (*cf* Proposition 2.2.8) that $\mathbb{C}_p$ is not isomorphic to $\mathbb{C}_p(\chi_{\text{cycl}})$ in the category $\text{Rep}_{\mathbb{C}_p}(G_{\mathbb{Q}_p})$.

1.4 Fontaine's strategy

We are now ready to explain the general principles of Fontaine's strategy for isolating the most interesting representations of the Galois group of a p-adic field and studying them. The material presented in this subsection comes from [21, Chap. II].

As before, let G be a topological group. Let also E be a fixed topological field. We consider a topological E-algebra B on which G acts and assume that the G-action on E is trivial. Under our assumptions, the category $\text{Rep}_E(G)$ is the category of E-linear representations of G.

Remark 1.4.1. In fact, in what follows, the topology on B will play no role since all the forthcoming definitions and results will be purely algebraic. Nevertheless, we prefer keeping the datum of the topology on B as it is more natural and all the rings B we shall consider later on will come equipped with a canonical topology.

The following definition is due to Fontaine.

Definition 1.4.2. Let $V \in \text{Rep}_E(G)$ be finite dimensional over E.
We say that V is *B-admissible* if the B-semi-linear representation $B \otimes_E V$ is trivial.

We denote by $\text{Rep}_E^{B\text{-adm}}(G)$ the full subcategory of $\text{Rep}_E(G)$ consisting of finite dimensional representations of E which are B-admissible. It is easy to check that $\text{Rep}_E^{B\text{-adm}}(G)$ is stable by direct sums, tensor products, and duals. Moreover the association $B \mapsto \text{Rep}_E^{B\text{-adm}}(G)$ is increasing in the following sense: any B_1-admissible representation is automatically B_2-admissible as soon as B_2 appears as an algebra over B_1.

Example 1.4.3. Let L be a finite extension of E. Take $G = \text{Gal}(L/E)$ and let it act naturally on L. Hilbert's theorem 90 (*cf* Theorem 1.3.3) shows that all finite dimension E-representation of G is L-admissible.

1.4.1 A criterium for B-admissibility

The aim of this paragraph is to establish a numerical criterium for recognizing B-admissible representations. In order to do so, we make the following assumptions[5] on the E-algebra B:

(H1) B is a domain,

(H2) $(\mathrm{Frac}\, B)^G = B^G$,

(H3) if $b \in B$, $b \neq 0$ and the E-line Eb is stable under G, then $b \in B^\times$.

It is easily seen that the assumption (H3) implies that B^G is a field. Indeed for any $b \in E$, $b \neq 0$, the line Eb is clearly stable under G. Thus b has to be invertible in B. Now we conclude by noticing that its inverse is also fixed by G. Moreover, by copying the proof of the second part of Theorem 1.3.3, one shows that the assumptions (H1) and (H2) ensure that the morphism $\alpha_W : B \otimes_{B^G} W^G \to W$ is injective for all $W \in \mathrm{Rep}_B(G)$ which are free of finite rank over B. In particular, this property holds true for W of the form $B \otimes_E V$ where V is finite dimensional E-linear representation of G.

Proposition 1.4.4. *We assume that B satisfies (H1), (H2) and (H3)*
Let $V \in \mathrm{Rep}_E(G)$ and set $W = B \otimes_E V$. We assume that V is finite dimensional over E. Then the following assertions are equivalent:

(i) W is trivial,

(ii) the morphism α_W is an isomorphism,

(iii) $\dim_{B^G} W^G = \dim_E V$.

Proof. Since B^G is a field, the equivalence between (i) and (ii) is obvious. Moreover the fact that (ii) implies (iii) is also clear. We then just have to prove that (iii) implies (ii).

We assume (iii) and denote by d the common dimension of V over E and W^G over B^G. The morphism $\alpha_W : B \otimes_{B^G} W^G \to B \otimes_E V$ is a B-linear morphism between two finite free B-modules of rank d. It is then enough to prove that its determinant is an isomorphism. Let $v_1, \dots, v_d$ be a E-basis of V and let $w_1, \dots, w_d$ be a B^G-basis of W^G. Let b be the unique element of B such that:

$$\alpha_W(v_1) \wedge \cdots \wedge \alpha_W(v_d) = b \cdot w_1 \wedge \cdots \wedge w_d. \tag{10}$$

From the injectivity of α_W, we derive $b \neq 0$. Let now $g \in G$. Applying g to (10), we get $gb = \eta \cdot b$ where η is defined by the identity $\alpha_W(gv_1) \wedge \cdots \wedge \alpha_W(gv_d) = \eta \cdot \alpha_W(v_1) \wedge \cdots \wedge \alpha_W(v_d)$. From the fact that the E-span of $v_1, \dots, v_d$ (which is V) is stable under the action of G, we deduce that η lies in E. Hence $gb \in Eb$. Consequently, the E-line Eb is stable by the action of G. Thanks to hypothesis (H3), we conclude that $b \in B^\times$ as wanted. □

Corollary 1.4.5. *Under the assumptions (H1), (H2) and (H3), the category $\mathrm{Rep}_E^{B\text{-adm}}(G)$ is stable by subobjects and quotients.*

Proof. Assume that we are given an exact sequence $0 \to V_1 \to V \to V_2 \to 0$ in the category $\mathrm{Rep}_E(G)$ and assume that V is B-admissible. Tensoring by B and taking the G-invariants, we obtain the exact sequence $0 \to (B \otimes_E V_1)^G \to (B \otimes_E V)^G \to (B \otimes_E V_2)^G$ from which we derive the inequality:

$$\dim_{B^G}(B \otimes_E V)^G \geq \dim_{B^G}(B \otimes_E V_1)^G + \dim_{B^G}(B \otimes_E V_2)^G. \tag{11}$$

Moreover we know that $\dim_{B^G}(B \otimes_E V_i)^G \leq \dim_E V_i$ for $i \in \{1, 2\}$. Therefore we get:

$$\dim_{B^G}(B \otimes_E V)^G \leq \dim_E V_1 + \dim_E V_2 = \dim_E V. \tag{12}$$

We know also that $\dim_{B^G}(B \otimes_E V)^G = \dim_E V$ thanks to the B-admissibility of V. Combining the inequalities (11) and (12), we find that $\dim_{B^G}(B \otimes_E V_i)^G$ has to be equal to $\dim_E V_i$, which proves that V_i (for $i \in \{1, 2\}$) is B-admissible. □

[5] Our assumptions are a bit stronger than Fontaine's ones. We chose these stronger hypothesis because they simplify the exposition and are sufficient for the applications we want to discuss here.

1.4.2 What's next?

Until now, we have spent a lot of time at defining a general abstract formalism whose main achievement is the notion of B-admissibility. This is certainly nice but still seems to be quite far from the applications. We devote this subsection to our readers who are impatient to connect the notion of B-admissibility to concrete properties of Galois representations and cohomology of algebraic varieties.

In the sequel, we will often use the locution *period rings* to refer to various rings B. This terminology is motivation by the role those rings B play in geometry (they often appear in comparison theorem between various cohomologies).

From now, we go back to the setting of §1.1. Precisely, we let K be a finite extension of $\mathbb{Q}_p$. We let $\bar{K}$ denote a fixed algebraic closure of K and we set $G_K = \text{Gal}(\bar{K}/K)$. Let $\mathbb{C}_p$ be the completion of $\bar{K}$. The action of G_K on $\bar{K}$ extends to a continuous action of G_K on $\mathbb{C}_p$. Finally, we recall that $\chi_{\text{cycl}} : G_K \to \mathbb{Z}_p^\times$ denotes the p-adic cyclotomic character of G_K.

$\mathbb{C}_p$-admissible representations. The first ring of periods we will consider is $\mathbb{C}_p$ itself, equipped with the p-adic topology and its natural action of G_K. The question of $\mathbb{C}_p$-admissibility of representations of G_K will be studied in details in §2; we will notably prove the following result (*cf* Theorem 2.2.1).

Theorem 1.4.6. *Let V be a $\mathbb{Q}_p$-linear finite dimensional representation of G_K. Then V is $\mathbb{C}_p$-admissible if and only if the inertia subgroup of G_K acts on V through a finite quotient.*

In other words, $\mathbb{C}_p$-admissibility detects those representations which are potentially unramified. This notion has then a strong arithmetical meaning.

Hodge–Tate representations. Theorem 1.4.6 shows that the notion of $\mathbb{C}_p$-admissibility is too strong and does not capture all interesting representations; for instance, the cyclotomic character is *not* $\mathbb{C}_p$-admissible. A larger class of representations is given by the notion of Hodge–Tate representations. By definition, a $\mathbb{Q}_p$-linear representation of G_K is Hodge–Tate if $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ decomposes as:

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} V = \mathbb{C}_p(\chi_{\text{cycl}}^{n_1}) \oplus \mathbb{C}_p(\chi_{\text{cycl}}^{n_2}) \oplus \cdots \oplus \mathbb{C}_p(\chi_{\text{cycl}}^{n_d}) \tag{13}$$

for some integers $n_1, \ldots, n_d$. The condition actually fits very well in the framework of B-admissibility as introduced above. Indeed, set $B_{\text{HT}} = \mathbb{C}_p[t, t^{-1}]$ (HT stands for Hodge–Tate) and let G_K act on it by the formula $g \cdot (at^i) = ga \cdot \chi_{\text{cycl}}(g)^i \cdot t^i$ for $g \in G_K$, $i \in \mathbb{Z}$ and $a \in \mathbb{C}_p$. One checks that V is Hodge–Tate if and only if it is B_{HT}-admissible. Besides, Theorem 1.4.6 is the starting point for studying Hodge–Tate representations. For example, it implies that the integers n_i's that appeared in (13) are uniquely determined up to permutation (*cf* Proposition 2.2.8). They are called the *Hodge–Tate weights* of the representation V.

Finally, Hodge-like decomposition theorems show that many representations coming from geometry are Hodge–Tate. This class of representations then seems particularly interesting.

De Rham and crystalline representations. Unfortunately, Hodge–Tate representations have several defaults. First, they are actually too numerous and, for this reason, it is difficult to describe them precisely and design tools to work with them efficiently. The second defect of Hodge–Tate representations is of geometric nature. Indeed, tensoring the étale cohomology with $\mathbb{C}_p$ (or equivalently, with B_{HT}) captures the *graded* module of the de Rham cohomology. However, it does not capture the entire complexity of de Rham cohomology, the point being that the de Rham filtration does not split canonically in the p-adic setting.

In order to work around this issues, Fontaine defined other period rings B "finer" than B_{HT}. The most classical period rings introduced by Fontaine are $B_{\text{crys}} \subset B_{\text{st}} \subset B_{\text{dR}}$; the corresponding admissible representations are called *crystalline*, *semi-stable* and *de Rham* respectively. Moreover, B_{dR} is a filtered field whose graded ring can be canonically identified with B_{HT}. This property, together with the aforementionned inclusions, imply the following implications:

$$\text{crystalline} \implies \text{semi-stable} \implies \text{de Rham} \implies \text{Hodge–Tate}.$$

In §3, we will discuss the construction of B_{dR} and B_{crys}, while the arithmetical and geometrical interest of these refined period rings will be presented in §4. Rapidly, let us say here that representations coming from the geometry, *i.e.* of the form $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$ where X is a smooth projective algebraic variety over $\mathbb{Q}_p$, are all de Rham. By definition, this means that the space $\left(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)\right)^{G_K}$ has the correct dimension. It turns out that this space has a very pleasant cohomological interpretation: it is canonically isomorphic to the de Rham cohomology of X, namely $H^r_{\mathrm{dR}}(X)$. We thus get an isomorphism:

$$B_{\mathrm{dR}} \otimes H^r_{\mathrm{dR}}(X) \xrightarrow{\sim} B_{\mathrm{dR}} \otimes H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$$

which is a the right p-adic analogue of the de Rham comparison theorem. Besides, the de Rham filtration on $H^r_{\mathrm{dR}}(X)$ can be rebuilt from the filtration on B_{dR}. This is the first apparition of the yoga of additional structures, which actually is ubiquitous in p-adic Hodge theory.

The introduction of B_{dR} resolves elegantly the geometric issue we have pointed out earlier. However, the class of B_{dR}-admissible representations is still rather large and not easy to describe. The ring B_{crys} is a subring of B_{dR} which is equipped with more structures and provides very powerful tools for describing crystalline representations. On the geometric side, crystalline representations correspond to the étale cohomology of varieties with good reduction and the space $\left(B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)\right)^{G_K}$ is related to the crystalline cohomology of (the special fibre of a proper smooth model of) X, equipped with its Frobenius endomorphism. All in all, we will obtain powerful methods for describing the étale cohomology of X with comparatively down-to-earth objects.

2 The first period ring: $\mathbb{C}_p$

After Tate and Fontaine's results on Hodge-like decompositions of cohomology, the first natural period ring to consider is $\mathbb{C}_p$ itself. In this section, we first study $\mathbb{C}_p$-admissibility and prove Theorem 1.4.6. The proof requires some preparation and occupies the first two subsections. The last subsection (§2.3) is devoted to expose Sen's theory, as developed in [35], whose aim is to go further than $\mathbb{C}_p$-admissibly and classify general $\mathbb{C}_p$-semi-linear representations.

Our approach differs a bit from usual presentations in that we will not use the langage of group cohomology but instead will keep working with (semi-linear) representations and vectors throughout the exposition.

2.1 Ramification in $\mathbb{Z}_p$-extensions

A first important ingredient in the proof of Theorem 1.4.6 is the study of ramification in Galois extensions over K whose Galois group is isomorphic to $\mathbb{Z}_p$.

Throughout this section, if L is an algebraic extension of K, we shall denote by $\mathcal{O}_L$ its ring of integers, by $\mathfrak{m}_L$ the maximal ideal of $\mathcal{O}_L$ and by $k_L = \mathcal{O}_L/\mathfrak{m}_L$ its residue field. If moreover L/K is finite, we shall denote by $v_L : L \to \mathbb{Z} \cup \{+\infty\}$ the valuation on L normalized by $v_L(L^\times) = \mathbb{Z}$. Set $e_L = v_L(p)$; it is the ramification index of the extension $L/\mathbb{Q}_p$. If F and L are two algebraic extensions of K with $F \subset L$ and $[L : F] < \infty$, we shall use the notation $e_{L/F}$ for the ramification index of L/F and the notation $\mathrm{Tr}_{L/F}$ for the trace map of L over F. When F is a finite extension of K, we have $e_{L/F} = \frac{e_L}{e_F}$.

In what follows, it is convenient to allow flexibility and work over a base F which is not necessarily K but a finite extension of it.

2.1.1 Higher ramification groups

We first recall briefly the theory of higher ramification groups as exposed in [39].

Let L/F be a finite Galois extension with Galois group G. For any nonnegative integer i, we define the i-th higher group of ramification of L/K as the subgroup G_i of G consisting of elements $g \in G$ such that g acts trivially on the quotient $\mathcal{O}_L/\mathfrak{m}_L^{i+1}$. One easily checks that the G_i's form a nonincreasing sequence of normal subgroups of G and that G_0 is the inertia subgroup of G. Besides, one proves that the quotient G_0/G_1 naturally embeds into $k_F^\times$ and thus is a cyclic of order prime to p, and, for $i > 0$, the quotient G_i/G_{i+1} embeds into $\mathfrak{m}_L^i/\mathfrak{m}_L^{i+1}$ and thus is a commutative p-group killed by p.

The ramification filtration we have just defined is not compatible with subextensions. However, we can recover some compatibility after a suitable reindexation. In order to do so, we first define the Herbrand function $\varphi_{L/F} : \mathbb{R}^+ \to \mathbb{R}^+$ by:

$$\varphi_{L/F}(u) = \frac{1}{e_{L/F}} \cdot \int_0^u \text{Card } G_t \cdot dt \tag{14}$$

where we agree that $G_t = G_{\lceil t \rceil}$ when t is a nonnegative real number and $\lceil \cdot \rceil$ is the ceiling function. Clearly $\varphi_{L/F}$ is increasing and defines a bijection from $\mathbb{R}^+$ to itself. Let $\psi_{L/F}$ be its inverse. For $u \in \mathbb{R}^+$, we set $G^u = G_{\psi_{L/F}(u)}$. One can then show the following property. If L_1 and L_2 are two finite Galois extensions of K with $L_1 \subset L_2$, then the canonical projection $\text{Gal}(L_2/F) \to \text{Gal}(L_1/F)$ maps surjectively $\text{Gal}(L_2/F)^u$ onto $\text{Gal}(L_1/F)^u$ for all $u \in \mathbb{R}^+$. This compatibility allows us the define ramification subgroups for infinite extensions: if L is a infinite Galois extension of K, we put $\text{Gal}(L/F)^u = \varprojlim_F \text{Gal}(L'/F)^u$ where L' runs over all finite extensions of F included in L.

Moreover the φ's and ψ's functions verify very pleasant composition formulae: if F, L_1 and L_2 are extensions of K as above, we have $\varphi_{L_2/F} = \varphi_{L_1/F} \circ \varphi_{L_2/L_1}$ and thus, passing to the inverse, $\psi_{L_2/F} = \psi_{L_2/L_1} \circ \psi_{L_1/F}$.

Finally, we observe that the knowledge of the upper numbering of the ramification filtration is equivalent to that of the lower numbering. Indeed the function $\psi_{L/F}$ can be recovered for the G^u's thanks to the formula:

$$\psi_{L/F}(t) = e_{L/F} \cdot \int_0^u \frac{du}{\text{Card } G^u}. \tag{15}$$

Now, taking the inverse of $\psi_{L/F}$, we reconstruct the function $\varphi_{L/F}$ and we can finally recover the lower numbering of the filtration ramification by letting $G_t = G^{\varphi_{L/F}(t)}$.

Relation with the different. We will often use the higher ramification groups in order to compute (or estimate) the different of the extension L/F. Let us first recall that the different $\mathcal{D}_{L/F}$ of L/F is the ideal of $\mathcal{O}_L$ characterized by the property that $\text{Tr}_{L/F}(x\mathcal{O}_L) \subset \mathcal{O}_F$ if and only if $x \in \mathcal{D}_{L/F}^{-1}$, *i.e.* $v_p(x) + v_p(\mathcal{D}_{L/F}) \geq 0$. Here we have introduced the valuation of an ideal, which is defined as the minimal valuation of one of its elements (or, equivalently, if it is generated by one element, the valuation of any generator). The following formula relates $\mathcal{D}_{L/F}$ with the size of the G_i's:

$$v_L(\mathcal{D}_{L/F}) = \sum_{i \geq 0} (\text{Card } G_i - 1).$$

From this relation, we derive easily the following formulae:

$$v_F(\mathcal{D}_{L/F}) = \lim_{t \to \infty} \left(\varphi_{L/F}(t) - \frac{t}{e_{L/F}} \right) = \lim_{u \to \infty} \left(u - \frac{\psi_{L/F}(u)}{e_{L/F}} \right). \tag{16}$$

Ramification and class field theory. When the extension L/F is finite and abelian, local class field theory gives a nice interpretation of the ramification filtration. More precisely, recall first that Artin reciprocity map provides an isomorphism $\text{Gal}(L/F) \simeq F^\times/N_{L/F}(L^\times)$ where $N_{L/F}$ is the norm of L over F. Under this isomorphism, the ramification subgroup $\text{Gal}(L/F)^u$ corresponds to the image in $F^\times/N_{L/F}(L^\times)$ of the congruence subgroup:

$$U_F^u = \left\{ x \in \mathcal{O}_F^\times \quad \text{s.t.} \quad x \equiv 1 \pmod{\mathfrak{m}_F^u} \right\} \subset F^\times.$$

A nontrivial consequence of this result is the Hasse–Arf theorem which states that the jumps of the filtration ramification in upper numbering (*i.e.* the real numbers u for which $G^{u+\varepsilon} \neq G^u$ for all $\varepsilon > 0$) are all integers.

2.1.2 The case of $\mathbb{Z}_p$-extensions

We now consider a Galois extension F_∞ of F. We assume that F_∞/F is ramified and that we are given an isomorphism $\alpha : \text{Gal}(F_\infty/F) \simeq \mathbb{Z}_p$. We remark that an extension with these properties always exists; it can be cooked up from the cyclotomic extension of F as discussed in §1.1.2.

For $r \geq 0$, let $\gamma_r = \alpha^{-1}(p^r) \in \text{Gal}(F_\infty/F)$ and F_r be the finite extension of F cut out by the closed subgroup generated by γ_r (that is the subgroup $\alpha^{-1}(p^r\mathbb{Z}_p)$). The F_r's then form a tower of extensions, in which each F_{r+1}/F_r is a cyclic extension of order p. More generally, if $s \geq r$, the extension F_s/F_r is cyclic of order p^{s-r} and its Galois group is generated by the class of γ_r.

Let also e_F be the absolute index of ramification of F defined as $e_F = v_F(p)$.

Proposition 2.1.1. *With the previous notations, there exists $a \in \mathbb{Z}$ such that, for u large enough,* $\text{Gal}(F_\infty/F)^u$ *is the closed subgroup generated by* $\gamma_{\lceil \frac{u-a}{e_F} \rceil}$.

Proof. For $u \in \mathbb{R}^+$, let $\rho(u)$ be the unique element r of $\mathbb{N} \cup \{+\infty\}$ for which $\text{Gal}(F_\infty/F)^u$ is topologically generated by γ_r. This definition yields a function $\rho : \mathbb{R}^+ \to \mathbb{N} \cup \{+\infty\}$ which is nondecreasing and left-continuous. The fact that F_∞/F is ramified shows that $\rho(0)$ is finite. By the Hasse–Arf theorem, the points of discontinuity of ρ are all integers. Moreover there must be infinitely many of them since the successive quotients of the ramification filtration are all killed by p. This implies that ρ takes finite values everywhere.

Let s be a positive integer. By local class field theory, we know that Artin's isomorphism $\text{Gal}(F_s/F) \simeq F^\times/N_{F_r/F}(F_r^\times)$ maps the subgroup $\text{Gal}(F_s/F)^u$ onto $U_F^u/(U_F^u \cap N_{F_r/F}(F_r^\times))$. Note that the group $\text{Gal}(F_s/F)^u$ is generated by the class of $\gamma_{\rho(u)}$. Its subgroup of p-th powers is then generated by $\gamma_{\rho(u)+1}$. On the other hand, a simple computation shows that the subgroup of p-th powers of U_F^u is equal to $U_F^{u+e_F}$ as soon as $u > \frac{e_F}{p-1}$. Comparing the subgroup of p-th powers of both sides, we obtain:

$$\min(s,\ \rho(u)+1) = \min(s,\ \rho(u+e_F))$$

whenever $u > \frac{e_F}{p-1}$. Letting s go to infinity, we end up with $\rho(u+e_F) = \rho(u)+1$ for $u > \frac{e_F}{p-1}$. This relation, combined with the facts that ρ is nondecreasing, left-continuous and takes integral values, implies that there exists a real constant a such that $\rho(u) = \lceil \frac{u-a}{e_F} \rceil$ for $u > \frac{e_F}{p-1}$. The fact that a is indeed an integer is a consequence of the Hasse–Arf theorem. ☐

Remark 2.1.2. Using formula (15), one can rephrase Proposition 2.1.1 as follows. For a positive integer r, let $\psi_r : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined by:

$$\begin{aligned}
\psi_r(u) &= u && \text{if } 0 \leq u < 1 \\
&= p(u-1)+1 && \text{if } 1 \leq u < 2 \\
&\ \ \vdots \\
&= p^{r-1}(u-r+1) + (1+p+\cdots+p^{r-2}) && \text{if } r-1 \leq u < r \\
&= p^r(u-r) + (1+p+\cdots+p^{r-1}) && \text{if } u \geq r
\end{aligned}$$

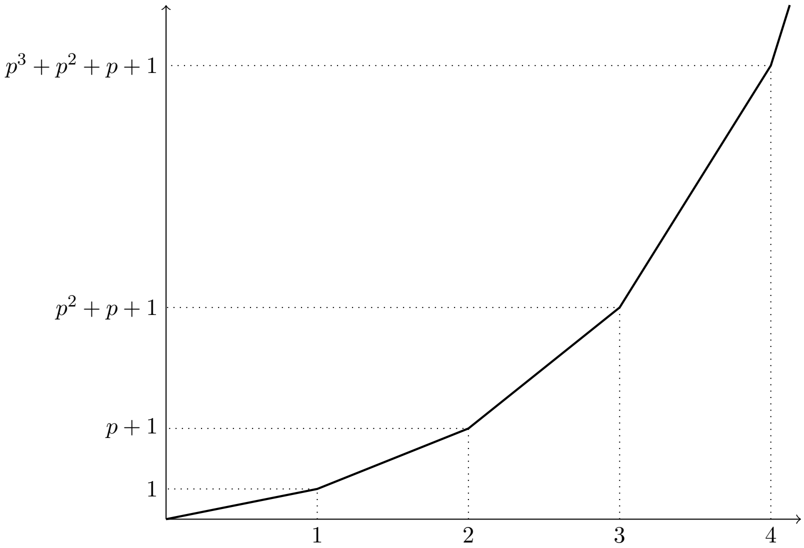

Figure 1: The graph of the function ψ_r ($r \geq 4$)

(*cf* Figure 1). Then, there exist $u_0 \in \mathbb{R}^+$ and two constants a and b such that $\psi_{F_r/F}(u) = e_F \cdot \psi_r(\frac{u-a}{e_F}) + b$ for all integer r and all $u \geq u_0$.

Proposition 2.1.1 has several interesting corollaries that we will derive below. We begin with two of them that give information about the behavior of the trace map. The first one (Proposition 2.1.3) concerns extensions living inside F_∞ and shows that traces in such extensions tend to decrease the norm by a large factor. On the contrary, the second one (Proposition 2.1.5) concerns extensions which are "orthogonal" to F_∞ and shows that traces in such extensions have a norm which is close to 1. The conceptual meaning of these results is that the extension F_∞/F captures almost all the ramification of $\bar{K}/F$.

Proposition 2.1.3. *There exists a constant c_1 (depending only on F and F_∞) for which the following property holds: for any positive integers r and s with $r \leq s$ and for any $x \in \mathcal{O}_{F_s}$, we have:*

$$v_p\big(\mathrm{Tr}_{F_s/F_r}(x)\big) \geq v_p(x) + s - r - c_1.$$

Proof. Fix a positive integer r. By the reformulation of Proposition 2.1.1 given in Remark 2.1.2, we have:

$$\frac{\psi_{F_r/F}(u)}{e_F p^r} = u - r - \frac{a}{e_F} + \frac{b}{p^r e_F} - \frac{p^r - 1}{p^r(p-1)}$$

when u is sufficiently large. Thanks to formula (16), we find:

$$v_p(\mathcal{D}_{F_r/F}) = r + \frac{a}{e_F} - \frac{b}{p^r e_F} + \frac{p^r - 1}{p^r(p-1)}.$$

Given now two integers r and s with $r \leq s$, the transitivity property of the different implies that:

$$v_p(\mathcal{D}_{F_s/F_r}) = v_p(\mathcal{D}_{F_s/F}) - v_p(\mathcal{D}_{F_r/F}) = s - r - \frac{b}{p^s e_F} + \frac{b}{p^r e_F} + \frac{p^s - 1}{p^s(p-1)} - \frac{p^r - 1}{p^s(p-1)}.$$

Then there exists $c \in \mathbb{N}$, not depending on r and s, such that $v_p(\mathcal{D}_{F_s/F_r}) \geq s - r - c$. Going back to the definition of the different, we obtain the inclusion $\mathrm{Tr}_{F_s/F_r}(p^{r-s+c}\mathcal{O}_{F_s}) \subset \mathcal{O}_{F_r}$. Let now $x \in F_s$ and let v be the integer part of $v_p(x)$. Then $p^{-v}x$ falls in $\mathcal{O}_{F_s}$, so that we get $\mathrm{Tr}_{F_s/F_r}(p^{r-s+c-v}x) \in \mathcal{O}_{F_r}$, *i.e.* $\mathrm{Tr}_{F_s/F_r}(x) \in p^{s-r-c+v}\mathcal{O}_{F_r}$. Consequently:

$$v_p(\mathrm{Tr}_{F_s/F_r}(x)) \geq s - r - c + v \geq v_p(x) + s - r - c - 1.$$

We can then take $c_1 = c + 1$. □

Remark 2.1.4. For a fixed integer r, we can glue the Tr_{K_s/K_r} (for s varying) and define a function $R_r : K_\infty \to K_r$ by $R_r(x) = p^{r-s}\,\mathrm{Tr}_{K_s/K_r}(x)$ for $x \in K_s$. We notice that the above definition makes sense because if $s \le t$, the functions $p^{r-s}\,\mathrm{Tr}_{K_s/K_r}$ and $p^{r-t}\,\mathrm{Tr}_{K_t/K_r}$ coincide on K_s. Proposition 2.1.3 shows that the function R_r obtained this way is uniformly continuous. It then extends (uniquely) to the completion $\hat{K}_\infty$ of K_∞.

The functions $R_r : \hat{K}_\infty \to K_r$ are called the *Tate's normalized traces*.

Proposition 2.1.5. *Let L be a finite Galois extension of F. For all $\varepsilon > 0$, there exist a positive integer r and an element $x \in \mathcal{O}_{L\cdot F_r}$ such that $v_p(\mathrm{Tr}_{L\cdot F_r/F_r}(x)) \le \varepsilon$.*

Proof. Up to replacing F by $F_\infty \cap L$, we may assume that F_∞ and L are linearly disjoint over F. We set $L_\infty = L\cdot K_\infty$ and $L_r = L\cdot F_r$ for all r. The extension L_∞/L is then a $\mathbb{Z}_p$-extension and the L_r's correspond to the subgroups $p^r\mathbb{Z}_p$.

By formula (16) and Remark 2.1.2, there exist $u_0 \in \mathbb{R}^+$ and $a, b, a', b' \in \mathbb{R}$ for which:

$$\begin{aligned}
\psi_{L/F}(u) &= e_{L/F}\cdot\big(u - v_F(\mathcal{D}_{L/F})\big)\\
\psi_{L_r/F_r}(u) &= e_{L/F}\cdot\big(u - v_{F_r}(\mathcal{D}_{L_r/F_r})\big)\\
\psi_{F_r/F}(u) &= e_F\cdot\psi_r\Big(\frac{u-a}{e_F}\Big) + b\\
\psi_{L_r/L}(u) &= e_F\cdot\psi_r\Big(\frac{u-a'}{e_L}\Big) + b'
\end{aligned}$$

for all $u \ge u_0$ and all positive integer r. Writing $\psi_{L_r/L}\circ\psi_{L/F} = \psi_{L_r/F_r}\circ\psi_{F_r/F}$, we obtain:

$$e_L\cdot\psi_r\Big(\frac{u}{e_F} - \frac{a}{e_F}\Big) + e_{L/F}\cdot\big(b - v_{F_r}(\mathcal{D}_{L_r/F_r})\big) = e_L\cdot\psi_r\Big(\frac{u}{e_F} - \frac{v_F(\mathcal{D}_{L/F})}{e_F} - \frac{a'}{e_L}\Big) + b'$$

for $u \ge u_0$. When r is sufficiently large, the above identity of functions implies, by comparing slopes, that $\frac{a}{e_F} = \frac{v_F(\mathcal{D}_{L/F})}{e_F} + \frac{a'}{e_L}$ and $b - v_{F_r}(\mathcal{D}_{L_r/F_r}) = b'$. From the latter equality, we derive $v_p(\mathcal{D}_{L_r/F_r}) = \frac{b-b'}{e_F p^r}$.

Let π_{F_r} be a uniformizer of F_r. Let y be an element of $\mathcal{D}_{L_r/F_r}$ with $v_p(y) = \frac{b-b'}{e_F p^r}$. By definition of the different, there exists $z \in \mathcal{O}_{L_r}$ such that $\mathrm{Tr}_{L_r/F_r}(\frac{z}{\pi_{F_r} y}) \notin \mathcal{O}_{F_r}$. In other words, $v_p(\mathrm{Tr}_{L_r/F_r}(\frac{z}{y})) < \frac{1}{e_F p^r}$. Set $n = \lceil b - b' \rceil$ and $x = \pi_{F_r}^n\,\frac{z}{y}$. We have $v_p(x) \ge \frac{n-(b-b')}{e_F p^r} \ge 0$; hence $x \in \mathcal{O}_{L_r}$. Moreover:

$$v_p(\mathrm{Tr}_{L_r/F_r}(x)) = v_p\big(\pi_{F_r}^n \mathrm{Tr}_{L_r/F_r}(\tfrac{z}{y})\big) = \frac{n}{e_F p^r} + v_p\big(\mathrm{Tr}_{L_r/F_r}(\tfrac{z}{y})\big) < \frac{n+1}{e_F p^r} < \frac{b-b'+2}{e_F p^r}.$$

We conclude the proof by noticing that, when r goes to infinity, the upper bound $\frac{b-b'+2}{e_F p^r}$ converges to 0. □

We shall also need the following result which is a refinement of the classical additive Hilbert's theorem 90, allowing in addition some control on the valuation.

Proposition 2.1.6. *There exists a constant c_2 (depending only on F and F_∞) for which the following property holds: for any positive integers r and s with $r \le s$ and for any $x \in \mathcal{O}_{F_s}$ with $\mathrm{Tr}_{F_s/F_r}(x) = 0$, there exists $y \in F_s$ such that (i) $\mathrm{Tr}_{F_s/F_r}(y) = 0$, (ii) $x = \gamma_r y - y$ and (iii) $v_p(y) \ge v_p(x) - c_2$.*

Moreover y is uniquely determined by the conditions (i) and (ii).

Proof. We set $d = s - r$ and:

$$y = -\frac{1}{p^d}\cdot\sum_{i=0}^{p^d-1} i\,\gamma_r^i(x).$$

Noticing that $\gamma_r^{p^d}$ is the identity on F_s, we find that $\gamma_r y - y = x + \frac{1}{p^d}\mathrm{Tr}_{F_s/F_r}(x) = x$. Moreover the assumption on the trace of x implies that $\mathrm{Tr}_{F_s/F_r}(y)$ vanishes as well.

Let now c_1 be the constant of Proposition 2.1.3 and set $c_2 = c_1 + 1$. For $m \in \{0, \ldots, d\}$, we define $x_m = \frac{1}{p^{d-m}}\mathrm{Tr}_{F_s/F_{r+m}}(x)$ and $y_m = -\frac{1}{p^m} \cdot \sum_{i=0}^{p^m-1} i\, \gamma_r^i(x_m)$. Obviously $x_d = x$ and $y_d = y$. Moreover, noticing that any integer between 0 and $p^m - 1$ can be uniquely written as $a + p^{m-1}b$ with $0 \leq a < p^{m-1}$ and $0 \leq b < p$, we obtain:

$$y_{m-1} - y_m \;=\; \frac{1}{p} \cdot \sum_{a=0}^{p^{m-1}-1} \sum_{b=0}^{p-1} b\, \gamma_r^{a+p^{m-1}b}(x_m).$$

Therefore $v_p(y_m) \geq \min(v_p(x_m) - 1,\, v_p(y_{m-1}))$ and so $v_p(y) = v_p(y_d) \geq \min_{1 \leq m \leq d} v_p(x_m) - 1$. By Proposition 2.1.3, we end up with $v_p(y) \geq v_p(x) - c_2$. The element y we have constructed then satisfies the requirements (i), (ii) and (iii).

It remains to prove unicity. Assume that we have given y_1 and y_2 such that $\mathrm{Tr}_{F_s/F_r}(y_1) = \mathrm{Tr}_{F_s/F_r}(y_2)$ and $x = \gamma_r y_1 - y_1 = \gamma_r y_2 - y_2$. Set $z = y_1 - y_2$. The second condition implies that z is fixed by γ_r. Hence $z \in F_r$ and $\mathrm{Tr}_{F_s/F_r}(z) = p^{s-r}z$. On the other hand, one has $\mathrm{Tr}_{F_s/F_r}(z) = \mathrm{Tr}_{F_s/F_r}(y_1) - \mathrm{Tr}_{F_s/F_r}(y_2) = 0$. We conclude that $p^{s-r}z = 0$ and hence $y_1 = y_2$. ☐

Remark 2.1.7. Using Tate's normalized traces (*cf* Remark 2.1.4), one may extend Proposition 2.1.6 for $x \in \hat{K}_\infty$. The result we obtain reads as follows: for all $x \in \hat{K}_\infty$ with $R_r(x) = 0$, there exists a unique $y \in K_r$ such that $x = \gamma_r y - y$ and $R_r(y) = 0$. Moreover, this element y satisfies $v_p(y) \geq v_p(x) - c_2$. In other terms, the function $(\gamma_r - \mathrm{id})$ is bijective on the kernel of R_r and its inverse is continuous.

2.2 $\mathbb{C}_p$-admissibility

We now come to the study of $\mathbb{C}_p$-admissibility of representations of G_K. The main objective of this subsection is to prove Theorem 1.4.6 whose statement is recalled below.

Theorem 2.2.1. *Let V be a $\mathbb{Q}_p$-linear finite dimensional representation of G_K. Then V is $\mathbb{C}_p$-admissible if and only if the inertia subgroup of G_K acts on V through a finite quotient.*

As an application, in §2.2.3, we will explain how Theorem 2.2.1 can be used to understand better the internal structure of Hodge–Tate representations.

2.2.1 Preliminaries

Before giving the proof of Theorem 2.2.1, we need some preparation. The first input that we shall use is Ax–Sen–Tate theorem, whose purpose is to compute the fixed subspace of $\mathbb{C}_p$ under the action of G_K. As in the previous subsection, we shall work over a base F which is itself a finite extension of K. For convenience, we set $G_F = \mathrm{Gal}(\bar{K}/F)$.

Theorem 2.2.2 (Ax–Sen–Tate). *We have $\mathbb{C}_p^{G_F} = F$.*

We shall prove a "finite" version of Ax–Sen–Tate theorem which is more precise.

Theorem 2.2.3. *There exists a constant c_3 (depending only on F) for which the following property holds: for all real number v and all $x \in \bar{K}$ such that $v_p(gx - x) \geq v$ for all $g \in G_F$, there exists $z \in K$ such that $v_p(x - z) \geq v - c_3$.*

Proof. Throughout the proof, we fix a $\mathbb{Z}_p$-extension F_∞ of F. We recall that such an extension always exists and can be built from the cyclotomic extension of F as discussed in §1.1.2.

Let L be a finite Galois extension of $\mathbb{Q}_p$ in which x lies. Thanks to Proposition 2.1.5, one can choose an integer r together with an element $\lambda \in L{\cdot}F_r$ with the property that $v_p(\mathrm{Tr}_{L\cdot F_r/F_r}(\lambda)) \leq 1$. We consider the elements:

$$y = \frac{\mathrm{Tr}_{L\cdot F_r/F_r}(\lambda x)}{\mathrm{Tr}_{L\cdot F_r/F_r}(\lambda)} \in F_r \quad \text{and} \quad z = \frac{1}{p^r} \cdot \mathrm{Tr}_{F_r/F}(y) \in F.$$

The fact that $v_p(gx - x) \geq v$ for all $g \in G_F$ implies that $v_p(y - x) \geq v - 1$.

Observe that $\mathrm{Tr}_{F_r/F}(y - z) = p^r z - p^r z = 0$ and $(\gamma_0 - \mathrm{id})(y - z) = \gamma_0 y - y$. In other words, the element $y - z$ has trace 0 and is an antecedent of $\gamma_0 y - y$ by the application $(\gamma_0 - \mathrm{id})$. By Proposition 2.1.6, it follows that $v_p(y - z) \geq v_p(\gamma_0 y - y) - c_2$. Now notice that the combinaison of $v_p(\gamma_0 x - x) \geq v$ and $v_p(y - x) \geq v - 1$ ensures that $v_p(\gamma_0 y - y) \geq v - 1$. Therefore, we obtain $v_p(y - z) \geq v - (c_2 + 1)$. Finally $v_p(x - z) \geq \min(v_p(x - y), v_p(y - z)) \geq v - (c_2 + 1)$ and we can take $c_3 = c_2 + 1$. □

Proof of Theorem 2.2.2. Let $x \in \mathbb{C}_p^{G_F}$. We consider a positive integer n. Since $\mathbb{C}_p$ is the completion of $\bar{\mathbb{Q}}_p$, one can find an element $x_n \in \bar{\mathbb{Q}}_p$ such that $v_p(x - x_n) \geq n$. We then have $v_p(gx_n - x_n) \geq n$ for all $g \in G_F$. By Theorem 2.2.3, one can find $z_n \in K$ such that $v_p(z_n - x_n) \geq n - c_3$. This implies that $v_p(z_n - x) \geq n - c_3$ as well. The sequence $(z_n)_{n\geq 1}$ then converges to x. Since $z_n \in K$ for all K, we obtain $x \in K$. □

Remark 2.2.4. As presented above, it seems that the proof of Theorem 2.2.2 uses class field theory (*via* Proposition 2.1.1). In fact, it is not the case because we have the choice on the $\mathbb{Z}_p$-extension F_∞. If we decide to take the $\mathbb{Z}_p$-part of the cyclotomic extension, the computation of the ramification filtration of $\mathrm{Gal}(F_\infty/F)$ can be carried out explicitely, so that Proposition 2.1.1 can be proved in this case without making any reference to class field theory.

The proof we have exposed above is essentialy due to Tate [40]. A few years later, Ax [2] reproves the theorem using a more direct and elementary argument. We presented Tate's proof because we believe that it serves as a very good introduction to the developments we will discuss afterwards, which are all modeled on the same strategy: in order to study the action of G_F (on some space), we will always first descend to F_∞ using Proposition 2.1.5 and then to F—or possibly only F_r for some finite r—using Proposition 2.1.3 or Proposition 2.1.6.

Ax's proof provides in addition an explicit value for the constant c_3, namely $\frac{p}{(p-1)^2}$. Ax asks for the optimality of this constant. In [34], Le Borgne answers this question and shows that the optimal constant is *not* $\frac{p}{(p-1)^2}$, but $\frac{1}{p-1}$. Le Borgne's proof follows Tate's strategy but uses a non Galois extension in place of the cyclotomic extension.

An extension of Hilbert's theorem 90. Another input we shall need is a variant of Hilbert's theorem 90 (*cf* Theorem 1.3.3) valid for *infinite* unramified extensions. We recall that K^{ur} denotes the maximal unramified extension of K (inside $\bar{K}$). We define $\hat{K}^{\mathrm{ur}}$ as the completion of K^{ur}; it is a field which naturally embeds into $\mathbb{C}_p$ and which is equipped with a canonical action of $\mathrm{Gal}(K^{\mathrm{ur}}/K)$.

Proposition 2.2.5. *Any finite dimensional $\hat{K}^{\mathrm{ur}}$-semi-linear representation of* $\mathrm{Gal}(K^{\mathrm{ur}}/K)$ *is trivial.*

Remark 2.2.6. Proposition 2.2.5 implies in particular that unramified representation of G_K are $\hat{K}^{\mathrm{ur}}$-admissible and then *a fortiori* $\mathbb{C}_p$-admissible. It then appears as a first step towards the proof of Theorem 2.2.1.

Proof of Proposition 2.2.5. In order to simplify notations, we denote by $\mathcal{O}$ the ring of integers of $\hat{K}^{\mathrm{ur}}$, and by $\mathfrak{m}$ its maximal ideal. We recall that the quotient $\mathcal{O}/\mathfrak{m}$ is isomorphic to an algebraic closure $\bar{k}$ of k. We recall also that $\mathrm{Gal}(K^{\mathrm{ur}}/K)$ is a procyclic group generated by the Frobenius $\mathrm{Frob}_q : x \mapsto x^q$ (where q is the cardinality of k).

Let W be a finite dimensional $\hat{K}^{\mathrm{ur}}$-semi-linear representation. We fix $(v_{1,0},\dots,v_{d,0})$ a basis of W over $\hat{K}^{\mathrm{ur}}$. Let $\mathcal{O}_W$ be a $\mathcal{O}$-span of $v_{1,0},\dots,v_{d,0}$. We are going to construct a sequence of tuples $(v_{1,n},\dots,v_{d,n})$ such that $v_{i,n+1} \equiv v_{i,n} \pmod{\mathfrak{m}^n}$ and $\mathrm{Frob}_q(v_{i,n}) \equiv v_{i,n} \pmod{\mathfrak{m}^n}$ for all $i \in \{1,\dots,d\}$ and all $n \in \mathbb{N}$.

We proceed by induction on n. The case $n = 1$ reduces to the fact that $\mathcal{O}_W/\mathfrak{m}\mathcal{O}_W$ is trivial as a $\bar{k}$-semi-linear representation of $\mathrm{Gal}(K^{\mathrm{ur}}/K) \simeq \mathrm{Gal}(\bar{k}/k)$. In order to prove this, we remark that, using continuity, $\mathcal{O}_W/\mathfrak{m}\mathcal{O}_W$ descends at finite level: there exist a finite extension ℓ of k and a ℓ-semi-linear representation W_ℓ of $\mathrm{Gal}(\ell/k)$ such that $\bar{k} \otimes_\ell W_\ell = \mathcal{O}_W/\mathfrak{m}\mathcal{O}_W$. The property we want to establish then follows from Hilbert's theorem 90 (*cf* Theorem 1.3.3).

We now assume that $(v_{1,n},\dots,v_{d,n})$ has been constructed. We look for vectors $w_1,\dots,w_n \in \mathcal{O}_W$ such that $\mathrm{Frob}_q(v_{i,n} + \pi^n w_i) \equiv v_{i,n} + \pi^n w_i \pmod{\mathfrak{m}^{n+1}}$ for all i. Letting $\bar{w}_i$ be the image of w_i in $\mathcal{O}_W/\mathfrak{m}\mathcal{O}_W$, the system we have to solve can be rewritten $\mathrm{Frob}_q\bar{w}_i - \bar{w}_i = \bar{c}_i$ $(1 \leq i \leq d)$ where $\bar{c}_i$ is defined as the image of $\frac{\mathrm{Frob}_q v_{i,n} - v_{i_n}}{\pi^n}$ in $\mathcal{O}_W/\mathfrak{m}\mathcal{O}_W$. It is then enough to prove that $(\mathrm{Frob}_q - \mathrm{id})$ is surjective on $\mathcal{O}_W/\mathfrak{m}\mathcal{O}_W$. This follows directly from the triviality of $\mathcal{O}_W/\mathfrak{m}\mathcal{O}_W$ and the fact that $(\mathrm{Frob}_q - \mathrm{id})$ is surjective on $\bar{k}$.

We conclude the proof by remarking that, for any fixed i, the sequence $v_{i,n}$ is Cauchy and hence converges to a vector $v_i \in \mathcal{O}_W$ on which $\mathrm{Gal}(K^{\mathrm{ur}}/K)$ acts trivially. Moreover the family of v_i's is an $\mathcal{O}$-basis of $\mathcal{O}_W$ (because its reduction modulo $\mathfrak{m}$ is a basis of $\mathcal{O}_W/\mathfrak{m}\mathcal{O}_W$) and then it is also a $\hat{K}^{\mathrm{ur}}$-basis of W. □

2.2.2 Proof of Theorem 2.2.1

We are now ready to prove Theorem 2.2.1.

Write $d = \dim_{\mathbb{Q}_p} V$. We first assume that the inertia subgroup acts on V through a finite quotient. In other words, there exists a finite extension L of K^{ur} for which $\mathrm{Gal}(\bar{K}/L)$ acts trivially on V. By Hilbert's theorem 90 (*cf* Theorem 1.3.3), the L-semi-linear representation $L \otimes_{\mathbb{Q}_p} V$ admits an L-basis $(v_1,\dots,v_d)$ on which the action of $\mathrm{Gal}(L/K^{\mathrm{ur}})$ is trivial. Consequently $\mathrm{Gal}(K^{\mathrm{ur}}/K)$ operates on the $\hat{K}^{\mathrm{ur}}$-span of $v_1,\dots,v_d$. By Proposition 2.2.5, this semi-linear representation is trivial. Therefore V is $(L{\cdot}\hat{K}^{\mathrm{ur}})$-admissible. It is then also $\mathbb{C}_p$-admissible.

We now focus on the converse. We assume that V is $\mathbb{C}_p$-admissible. Then by definition, there exists a $\mathbb{C}_p$-basis $(w_1,\dots,w_d)$ of $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ with the property that $gw_i = w_i$ for all $g \in G_K$ and all $i \in \{1,\dots,d\}$. Let $(v_1,\dots,v_d)$ be a basis of V over $\mathbb{Q}_p$ and let $P \in \mathrm{GL}_d(\mathbb{C}_p)$ be the matrix representating the change of basis between the v_i's and the w_i's. Up to rescaling the v_i's, we may assume without loss of generality that $P \in M_d(\mathcal{O}_{\mathbb{C}_p})$.

From the fact that the $\mathbb{Q}_p$-span of the v_i's is stable under the action of G_K, we derive that the matrix $U_g = P^{-1} \cdot gP$ has coefficients in $\mathbb{Q}_p$ for all $g \in G_K$. Let c_3 be the constant of Theorem 2.2.3 and let v be a positive integer for which $p^v{\cdot}P^{-1}$ has coefficients in $\mathcal{O}_{\mathbb{C}_p}$. By continuity of the action of G_K, denoting by I_d the identity matrix of size d, there exists an open subgroup H of G_K such that $v_p(U_g - I_d) \geq v + c_3 + 1$ for all $g \in H$. Multiplying by P on the left, we get $v_p(P - gP) \geq v + c_3 + 1$ for all $g \in H$. Applying now Theorem 2.2.3 (to each entry of P), we find a matrix $P_0 \in \mathrm{GL}_n(L)$ such that $P \equiv P_0 \pmod{p^{v+1}}$. multiplying by P^{-1} on the left, we get $P^{-1}P_0 \equiv I_d \pmod{p}$. Define $M = P^{-1}P_0$. Writing $P = P_0M^{-1}$, we find $M \cdot gM^{-1} = U_g$ for all $g \in H$. Since $M \equiv I_d \pmod{p}$, the matrix $N = \log M$ is well defined and satisfies the relation $N - gN = \log U_g$ for all $g \in H$.

Let F be the extension of K cut out by H. We are going to prove that $H \cap I_K$ operates trivially on N. Let ξ be an entry of N. The relation $N - gN = \log U_g$ ensures that $\xi - g\xi \in \mathbb{Z}_p$ whenever g is in H. Define the function $\alpha : H \to \mathbb{Z}_p$ by $\alpha(g) = g\xi - \xi$. The computation

$$\alpha(g_1g_2) = g_1g_2\xi - \xi = g_1(g_2\xi - \xi) + (g_1\xi - \xi) = (g_2\xi - \xi) + (g_1\xi - \xi) = \alpha(g_2) + \alpha(g_1)$$

shows that α is an additive character. Its kernel defines a Galois extension F_∞ of F whose Galois group embeds into $\mathbb{Z}_p$. Moreover, by construction, $H \cap \ker\alpha$ acts trivially on ξ. We then need to prove that F_∞ is unramified over F.

We assume by contraction that the extension F_∞/F is ramified. In particular, it is not trivial, and hence it is a $\mathbb{Z}_p$-extension. Proposition 2.1.3 then applies and ensures that there exists a constant c_1 such that:

$$v_p(\mathrm{Tr}_{F_s/F_r}(z)) \geq v_p(z) + s - r - c_1 \tag{17}$$

whenever $s \geq r$ and $z \in F_s$. Let v be a positive real number and let x be an element of $\bar{K}$ such that $v_p(x - \xi) \geq v$. From the equality $g\xi - \xi = \alpha(g)$, we derive $v_p\big(gx - x - \alpha(g)\big) \geq v$ for all $g \in H$. In particular, if $g \in H \cap \ker \alpha$, we obtain $v_p(gx - x) \geq v$. By continuity, this estimation is also correct for $g \in \mathrm{Gal}(\bar{K}/F_s)$ for some integer s. Repeating the first part of the proof of Theorem 2.2.3 (with the $\mathbb{Z}_p$-extension F_∞/F) and possibly enlarging s, we find that there exists $y \in F_s$ with the property that $v_p(x - y) \geq v - 1$. Thus $v_p(\xi - y) \geq v - 1$ as well.

Fix now $g \in H$ and set $z = gy - y - \alpha(g)$. By our assumption on ξ, we know that $v_p(z) \geq v - 1$. Using (17) with $r = 0$, we obtain $v_p(\mathrm{Tr}_{F_s/F_0}(z)) \geq v + s - c_1$. On the other hand, a direct computations yields $\mathrm{Tr}_{F_s/F_0}(z) = -p^s\alpha(g)$. Combining these two inputs, we deduce $v_p(\alpha(g)) \geq v - c_1$. Since this estimation holds for all $g \in H$ and all $v \in \mathbb{R}^+$, we end up with $\alpha = 0$. This means that $F_\infty = F$ and then contradicts our assumption that F_∞/F was ramified.

Remark 2.2.7. It follows from the proof above (*cf* in particular the first paragraph of the proof) that a representation is $\mathbb{C}_p$-admissible if and only if it is $(L{\cdot}\hat{K}^{\mathrm{ur}})$-admissible for a *finite* extension L of K^{ur}. We will reuse this property in §4.2 when we will compare $\mathbb{C}_p$-representations with de Rham representations.

2.2.3 Application to Hodge–Tate representations

Beyond its obvious own interest, Theorem 2.2.1 can be thought of as a first result towards the study of Hodge–Tate representations. Recall that a finite dimensional $\mathbb{Q}_p$-linear representation V of G_K is Hodge–Tate if $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ decomposes as:

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} V = \mathbb{C}_p(\chi_{\mathrm{cycl}}^{n_1}) \oplus \mathbb{C}_p(\chi_{\mathrm{cycl}}^{n_2}) \oplus \cdots \oplus \mathbb{C}_p(\chi_{\mathrm{cycl}}^{n_d}) \tag{18}$$

for some integers n_i's. Theorem 2.2.1 implies the following unicity result.

Proposition 2.2.8. *Let V be a finite dimensional Hodge–Tate representation of G_K. Then the integers n_i of Eq.* (18) *are uniquely determined up to permutation.*

Proof. We have to show that, if $\mathbb{C}_p(\chi_{\mathrm{cycl}}^{n_1}) \oplus \cdots \oplus \mathbb{C}_p(\chi_{\mathrm{cycl}}^{n_d}) \simeq \mathbb{C}_p(\chi_{\mathrm{cycl}}^{m_1}) \oplus \cdots \oplus \mathbb{C}_p(\chi_{\mathrm{cycl}}^{m_{d'}})$, then $d = d'$ and the n_i's agree with the m_i's up to permutation. For this, it is enough to check that, given two integers n and m,

$$\mathrm{Hom}_{\mathrm{Rep}_{\mathbb{C}_p}(G_K)}\big(\mathbb{C}_p(\chi_{\mathrm{cycl}}^{n}), \mathbb{C}_p(\chi_{\mathrm{cycl}}^{m})\big) \tag{19}$$

is a one dimensional K-vector space if $n = m$, and is zero otherwise.

Let $W = \mathrm{Hom}_{\mathbb{C}_p}\big(\mathbb{C}_p(\chi_{\mathrm{cycl}}^{n}), \mathbb{C}_p(\chi_{\mathrm{cycl}}^{m})\big) \simeq \mathbb{C}_p(\chi_{\mathrm{cycl}}^{n-m})$ (equipped with its Galois action). The space (19) is equal to W^{G_K}. When $n = m$, it is then $\mathbb{C}_p^{G_K}$ which is indeed equal to K by Ax–Sen–Tate theorem (*cf* Theorem 2.2.2). If $n \neq m$, we need to prove that W is not trivial, which means that the representation $V = \mathbb{Q}_p(\chi_{\mathrm{cycl}}^{n-m})$ is not $\mathbb{C}_p$-admissible. By Theorem 2.2.1, we are reduced to justify that the inertia subgroup of $\mathbb{Q}_p$ does not act on V through a finite quotient. This is clear because the extension cut out by the kernel of $\chi_{\mathrm{cycl}}^{n-m}$ is the p-adic cyclotomic extension which is infinitely ramified. □

Remark 2.2.9. A byproduct of the proof above is that the $\mathbb{C}_p$-semi-linear representation $\mathbb{C}_p(\chi_{\mathrm{cycl}}^{n})$ has no nonzero invariant vector when $n \neq 0$. We will reuse repeatedly this property in the sequel.

Example 2.2.10. Recall that, assuming $p > 2$, we have classified the characters of $G_{\mathbb{Q}_p}$ in Proposition 1.1.1: they are all of the form $\mu_\lambda \cdot \chi_{\text{cycl}}^a \cdot \omega_{\text{cycl}}^b$ with $a \in \mathbb{Z}_p$ and $b \in \mathbb{Z}/(p-1)\mathbb{Z}$. Here μ_λ denotes the unramified character taking the Frobenius Frob_q to λ and $\omega_{\text{cycl}} = [\chi_{\text{cycl}} \bmod p]$. Since the representations $\mathbb{C}_p(\mu_\lambda)$ and $\mathbb{C}_p(\omega_{\text{cycl}}^b)$ are $\mathbb{C}_p$-admissible, we obtain:

$$\mathbb{C}_p(\mu_\lambda \cdot \chi_{\text{cycl}}^a \cdot \omega_{\text{cycl}}^b) \simeq \mathbb{C}_p(\chi_{\text{cycl}}^a).$$

Hence the character $\mu_\lambda \cdot \chi_{\text{cycl}}^a \cdot \omega_{\text{cycl}}^b$ is Hodge–Tate if and only if $a \in \mathbb{Z}$. In this case, its Hodge–Tate weight is a.

Example 2.2.11. Let $\alpha : G_K \to \mathbb{Z}_p$ be an additive character, *e.g.* $\alpha = \log \chi_{\text{cycl}}$. Consider the two dimensional representation V corresponding to the group homomorphism:

$$G_K \to \text{GL}_2(\mathbb{Q}_p), \quad g \mapsto \begin{pmatrix} 1 & \alpha(g) \\ 0 & 1 \end{pmatrix}.$$

In order terms, $V = \mathbb{Q}_p^2$ and G_K acts to V by $g \cdot (u, v) = (u,\, v + \alpha(g)u)$. We have an obvious exact sequence $0 \to \mathbb{Q}_p \to V \to \mathbb{Q}_p \to 0$ where the action of G_K on the two copies of $\mathbb{Q}_p$ is the trivial action. Tensoring this sequence by $\mathbb{C}_p$, we get $0 \to \mathbb{C}_p \to V \to \mathbb{C}_p \to 0$. The representation V is Hodge–Tate if and only if the above sequence splits, if and only if V is $\mathbb{C}_p$-admissible. By Theorem 2.2.1, this happens if and only if $\alpha(I_K)$ is finite (where I_K is the inertia subgroup of G_K). Since $\alpha(I_K)$ is a subgroup of $\mathbb{Z}_p$, the previous condition is equivalent to the fact that $\alpha(I_K)$ is reduced to 0. As a conclusion, the representation V is Hodge–Tate if and only if α is unramified. In this case, the Hodge–Tate weights of V are 0 with multiplicity 2.

Hodge–Tate representations and admissibility. It is important to notice that the class of Hodge–Tate representations fits very well in Fontaine's framework presented in §1.4. Precisely, let us consider the rings $B_{\text{HT}} = \mathbb{C}_p[t, t^{-1}]$ and $B'_{\text{HT}} = \mathbb{C}_p((t))$. We equip them with the Galois action obtained by letting G_K act naturally on $\mathbb{C}_p$ and act on t by $gt = \chi_{\text{cycl}}(g)\, t$ for all $g \in G_K$. In addition, we define a filtration of B'_{HT} by $\text{Fil}^m B'_{\text{HT}} = t^m \mathbb{C}_p[[t]]$ for m varying in $\mathbb{Z}$. The graded ring of B'_{HT} is, by definition:

$$\text{gr}\, B'_{\text{HT}} = \bigoplus_{m \in \mathbb{Z}} \text{Fil}^m B'_{\text{HT}} / \text{Fil}^m B'_{\text{HT}}.$$

We observe that it is canonically isomorphic to B_{HT}. Besides, we have a natural G_K-equivariant inclusion $B_{\text{HT}} \to B'_{\text{HT}}$. Ax–Sen–Tate theorem, together with the fact that $\mathbb{C}_p(\chi_{\text{cycl}}^n)$ has no nonzero invariant vectors as soon as $n \neq 0$, implies that $(B_{\text{HT}})^{G_K} = (B'_{\text{HT}})^{G_K} = K$.

Proposition 2.2.12. *The rings B_{HT} and B'_{HT} satisfy Fontaine's assumptions (H1), (H2) and (H3) (introduced in §1.4.1).*

Proof. This is obvious for B'_{HT} since it is a field. As for B_{HT}, it is clearly a domain. Moreover since B'_{HT} is a field, we have $B_{\text{HT}} \subset \text{Frac}\, B_{\text{HT}} \subset B'_{\text{HT}}$. Taking the G_K-invariants, we obtain $(\text{Frac}\, B_{\text{HT}})^{G_K} = K$; hence B_{HT} satisfies (H2). Finally, we prove that B_{HT} satisfies (H3). Let $x \in B_{\text{HT}}$, $x \neq 0$ and assume that the line $\mathbb{Q}_p x$ is stable by G_K. We have to prove that x is invertible in B_{HT}. Up to multiplying x by some power of t, we may assume that $x \in \mathbb{C}_p[t]$. Write $x = a_0 + a_1 t + \cdots + a_n t^n$ where the a_i's are in $\mathbb{C}_p$. Our assumption implies that there exists $\lambda \in \mathbb{Q}_p$ such that $g a_i \cdot \chi(g)^i = \lambda a_i$ for all $g \in G_K$ and all $i \in \{0, 1, \dots, n\}$. Let j be an index for which $a_j \neq 0$ and write $\mu_i = \frac{a_i}{a_j}$ for all i. We then have $g\mu_i = \chi(g)^{j-i} \mu_i$ for all g and i. If $i \neq j$, this implies that $\mu_i = 0$ since $\mathbb{C}_p(\chi_{\text{cycl}}^{j-i})^{G_K} = 0$. Therefore x has to be equal to $a_j t^j$, and so is invertible in B_{HT}. □

Proposition 2.2.13. *Let V be a finite dimensional $\mathbb{Q}_p$-linear representation. Then V is Hodge–Tate if and only if it is B_{HT}-admissible, if and only if it is B'_{HT}-admissible.*

Proof. Write $d = \dim_{\mathbb{Q}_p} V$. Observe that $B_{\mathrm{HT}} = \bigoplus_{m\in\mathbb{Z}} \mathbb{C}_p(\chi_{\mathrm{cycl}}^m)$ as a $\mathbb{C}_p$-semi-linear representation. Therefore:

$$\left(V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}}\right)^{G_K} \simeq \bigoplus_{m\in\mathbb{Z}} \left(V \otimes \mathbb{C}_p(\chi_{\mathrm{cycl}}^m)\right)^{G_K}.$$

Suppose that V is Hodge–Tate. Let $m_1, \ldots, m_s$ be its Hodge–Tate weights and $e_1, \ldots, e_s$ be the corresponding multiplicities. The space $\left(V \otimes \mathbb{C}_p(\chi_{\mathrm{cycl}}^{-m_i})\right)^{G_K}$ has then dimension e_i. Summing up all these contributions, we find that $(V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}})^{G_K}$ has dimension d, which means that V is B_{HT}-admissible.

The converse and the case of B'_{HT} are proved in a similar fashion and left to the reader. □

2.3 Complement: Sen's theory

The aim of this subsection is to expose Sen's theory [35] whose objective is to provide a systematic study of finite dimensional $\mathbb{C}_p$-semi-linear representations of G_K. In what follows, we choose and fix once for all a $\mathbb{Z}_p$-extension K_∞ of K. We let $\alpha : \mathrm{Gal}(K_\infty/K) \to \mathbb{Z}_p$ be the attached group isomorphism. As in §2.1.2, we put $\gamma_r = \alpha^{-1}(p^r)$ and let K_r be the subextension of K_∞ corresponding to the closed subgroup $\alpha^{-1}(p^r\mathbb{Z}_p)$.

We recall that one possible choice is $\alpha = \log \chi_{\mathrm{cycl}}$, in which case K_∞ is the $\mathbb{Z}_p$-part of the cyclotomic extension of K (*cf* §1.1.2). Actually, strictly speaking, Sen's theory only concerns this particular choice of α. However the extension of general α's is straightforward. In what follows, we do *not* restrict ourselves to $\alpha = \log \chi_{\mathrm{cycl}}$.

Remark 2.3.1. Recently, Berger and Colmez [4] generalized Sen's theory, allowing α to take its values in any p-adic Lie group (possibly noncommutative). Their theory relies on the notion of locally analytic vectors, which is not needed in classical Sen's theory (finite vectors are enough as we shall explain below). We will not expose their generalization in the article and do restrict ourselves to homomorphisms α taking their values in $\mathbb{Z}_p$.

We recall that, given a topological group G and a topological ring B on which G acts, we have introduced the notation $\mathrm{Rep}_B(G)$ for the category of B-semi-linear representations of G. Let $\mathrm{Rep}^{\mathrm{f}}_B(G)$ denote the full subcategory of $\mathrm{Rep}_B(G)$ consisting of representations which are finitely generated as a B-module. When B is the field, $\mathrm{Rep}^{\mathrm{f}}_B(G)$ is then the category of *finite dimensional* B-semi-linear representations of G.

Descend. The first result towards Sen's theory is Proposition 2.3.2 just below, which could understood as an analogue of Hilbert's theorem 90 for the Galois group $\mathrm{Gal}(\bar{K}/K_\infty)$.

Proposition 2.3.2. *Let $W \in \mathrm{Rep}^{\mathrm{f}}_{\mathbb{C}_p}(G_K)$. Then there exist an integer r and a $\mathbb{C}_p$-basis $v_1, \ldots, v_d$ of W such that $gv_i = v_i$ for all $g \in \mathrm{Gal}(\bar{K}/K_\infty)$.*

Proof. The proof is similar to that of Proposition 2.2.5.

Let $\mathcal{O}_W$ be any $\mathcal{O}_{\mathbb{C}_p}$-lattice in W. As a first step, we are going to construct a $\mathbb{C}_p$-basis $w_1, \ldots, w_d$ of W with $w_i \in \mathcal{O}_W$, $gw_i \equiv w_i \pmod{p^2\mathcal{O}_W}$ for all $g \in \mathrm{Gal}(\bar{K}/K_\infty)$ and $p\mathcal{O}_W \subset \mathcal{O}_{\mathbb{C}_p} w_1 \oplus \cdots \oplus \mathcal{O}_{\mathbb{C}_p} w_d$. By continuity of the Galois action, there exists a finite Galois extension L of K such that $gw \equiv w \pmod{p^2\mathcal{O}_W}$ for all $g \in \mathrm{Gal}(\bar{K}/L)$ and all $w \in W$. For a positive integer r, set $L_r = L{\cdot}K_r$. By the proof of Proposition 2.1.3, we know that there exists r for which $v_p(\mathcal{D}_{L_r/K_r}) < e_{L/K}^{-1}$. We fix such an r.

Let $\lambda_1, \ldots, \lambda_m$ be a $\mathcal{O}_{K_r}$-basis of $\mathcal{O}_{L_r}$ and let $g_1, \ldots, g_m$ be the elements of $\mathrm{Gal}(L_r/K_r)$. For $i \in \{1, \ldots, m\}$, choose $\hat{g}_i \in G_K$ a lift of g_i. We define the elements:

$$y_{i,j} = \sum_{i'=1}^{m} \hat{g}_{i'}(\lambda_j x_i) = \sum_{i'=1}^{m} g_{i'}(\lambda_j) \cdot \hat{g}_{i'}(x_i)$$

for i varying between 1 and d and j varying between 1 and m. It is easily seen that $gy_{i,j} \equiv y_{i,j} \pmod{p^2\mathcal{O}_W}$ for all $g \in \text{Gal}(\bar{K}/K_r)$. Moreover, the determinant of the matrix $(g_i(\lambda_j))_{1 \leq i,j \leq m}$ is, by definition, the discriminant of L_r/K_r. Its p-adic valuation is then less than 1 thanks to our assumption on $v_p(\mathcal{D}_{L_r/K_r})$. We deduce that there exist $\mu_1, \ldots, \mu_m \in \mathcal{O}_{L_r}$ with the property that $\sum_{j=1}^m \mu_j \text{Tr}_{L_r/K_r}(\lambda_j) = p$. Hence $\sum_{j=1}^m \mu_j y_{i,j} \equiv px_i \pmod{p^2\mathcal{O}_W}$ for all i. The $\mathcal{O}_{\mathbb{C}_p}$-span of the $y_{i,j}$'s then contains $p\mathcal{O}_W$. Among these vectors, one can select d of them $w_1, \ldots, w_d$ whose span still contains $p\mathcal{O}_W$. The w_i's satisfy all the announced properties.

The second step of the proof consists in lifting the w_i's by a process of successive approximations. In order to simplify the notations, we redefine $\mathcal{O}_W$ as the $\mathcal{O}_{\mathbb{C}_p}$-span of $w_1, \ldots, w_d$. With the new definition, we have $gw_i \equiv w_i \pmod{p\mathcal{O}_W}$ for all $g \in \text{Gal}(\bar{K}/K_\infty)$. We will construct by induction on n a sequence of families $(v_{1,n}, \ldots, v_{d,n})$ satisfying the following congruences:

$$v_{i,n+1} \equiv v_{i,n} \pmod{p^n\mathcal{O}_W} \quad \text{and} \quad gv_{i,n} \equiv v_{i,n} \pmod{p^n\mathcal{O}_W}$$

for all $i \in \{1, \ldots, d\}$, $n \in \mathbb{N}$ and $g \in \text{Gal}(\bar{K}/K_\infty)$. For $n = 1$, we set $v_{i,1} = w_i$. Now we assume that the $v_{i,n}$'s have been constructed. By continuity there exists a finite Galois extension L of K such that $gv_{i,n} \equiv v_{i,n} \pmod{p^{n+2}\mathcal{O}_W}$ for all $g \in \text{Gal}(\bar{K}/L)$. By Proposition 2.1.3, there exist an integer r and $\lambda \in \mathcal{O}_{L_r}$ (with $L_r = L{\cdot}K_r$) such that $v_p(\text{Tr}_{L_r/K_r}(\lambda)) \leq 1$. As in the first step, we let $g_1, \ldots, g_m$ be the elements of $\text{Gal}(L_r/K_r)$ and we choose a lifting $\hat{g}_i \in G_K$ of g_i. We define:

$$v_{i,n+1} = \frac{1}{\text{Tr}_{L_r/K_r}(\lambda)} \cdot \sum_{j=1}^m \hat{g}_j(\lambda v_{i,n}) = \frac{1}{\text{Tr}_{L_r/K_r}(\lambda)} \cdot \sum_{j=1}^m g_j(\lambda) \cdot \hat{g}_j(v_{i,n})$$

and check that the $v_{i,n+1}$'s satisfy the desired requirements.

We conclude the proof by taking the limit with respect to n. □

Proposition 2.3.2 tells us that the W is trivial when viewed as a $\mathbb{C}_p$-linear representation of $\text{Gal}(\bar{K}/K_\infty)$. Moreover by the proof of Ax–Sen–Tate theorem, the fixed field $\mathbb{C}_p^{\text{Gal}(\bar{K}/K_\infty)}$ is the completion of K_∞, that we shall call $\hat{K}_\infty$. By general results of trivial semi-linear representations (*cf* §1.3.2), we then have an isomorphism

$$\mathbb{C}_p \otimes_{\hat{K}_\infty} W^{\text{Gal}(\bar{K}/K_\infty)} \simeq W$$

for all $\mathbb{C}_p$-semi-linear representation of W. We notice that $W^{\text{Gal}(\bar{K}/K_\infty)}$ inherits an action of $\text{Gal}(K_\infty/K)$.

Finite vectors. Set $\Gamma = \text{Gal}(K_\infty/K)$. The second step is Sen's theory is the study of $\hat{K}_\infty$-semi-linear representations of Γ. To this attempt, Sen defines the subspace of *finite* vectors as follows.

Definition 2.3.3. Let $W \in \text{Rep}^{\text{f}}_{\hat{K}_\infty}(\Gamma)$. A vector $v \in W$ is *finite* if the K_∞-subspace of W generated by the gv for g varying in Γ is finite dimensional over K_∞.

As an example, the subspace of finite vectors of the semi-linear representation $\hat{K}_\infty$ itself is K_∞. In general, one easily checks that the subspace of finite vectors is a vector space over K_∞.

Proposition 2.3.4. *Let $W \in \text{Rep}^{\text{f}}_{\hat{K}_\infty}(\Gamma)$. Then, there exist an integer r and a basis $(v_1, \ldots, v_d)$ of W with the property that the K_r-span of the v_i's is stable under the Γ-action.*

Remark 2.3.5. Obviously, the v_i's of Proposition 2.3.4 are finite in the sense of Definition 2.3.3. Therefore, we deduce that the subspace of finite vectors of W generates W as a $\hat{K}_\infty$-vector space. Finite vectors are then numerous.

Proof of Proposition 2.3.4. Let c_2 be the constant of Proposition 2.1.6. It is harmless to assume that c_2 is an integer. To simplify notation, we write $L = \hat{K}_\infty$. Let $\mathcal{O}_L$ be the ring of integers of L. We choose a $\mathcal{O}_L$-lattice $\mathcal{O}_W$ inside W. By continuity, there exists an integer r such that $gw \equiv w \pmod{p^{c_2+1}\mathcal{O}_W}$ for all $g \in \mathrm{Gal}(K_\infty/K_r)$ and all $w \in W$. We choose and fix such an r. The group $\mathrm{Gal}(K_\infty/K_r)$ acts on $\mathcal{O}_W$ and on all the quotients $\mathcal{O}_W/p^n\mathcal{O}_W$ for $n \in \mathbb{N}$.

We are going to construct, by induction of n, a sequence of families $(v_{1,n}, \ldots, v_{d,n})_{n\geq 1}$ of elements of $\mathcal{O}_W$ with the following properties:

(i) for all n, the family $v_{1,n}, \ldots, v_{d,n}$ is an $\mathcal{O}_L$-basis of $\mathcal{O}_W$,

(ii) for all $n \geq 1$ and all i, $v_{i,n+1} \equiv v_{i,n} \pmod{p^n\mathcal{O}_W}$

(iii) the $\mathcal{O}_{K_r}$-submodule of $\mathcal{O}_W/p^{n+c_2}\mathcal{O}_W$ generated by the classes of the $v_{i,n}$'s ($1 \leq i \leq d$) is stable under the $\mathrm{Gal}(K_\infty/K_r)$-action.

For $n = 1$, we pick an arbitrary $\mathcal{O}_L$-basis $v_{1,1}, \ldots, v_{d,1}$ of $\mathcal{O}_W$. Since $\mathrm{Gal}(K_\infty/K_r)$ acts trivially on $\mathcal{O}_W/p^{c_2+1}\mathcal{O}_W$, all the requirements are fulfilled. We now assume that $v_{1,n}, \ldots, v_{d,n}$ have been constructed. By the induction hypothesis, for all i, we can write $\gamma_r v_{i,n} = v_{i,n} + \sum_{j=1}^{d}(\lambda_{i,j} + \varepsilon_{i,j})v_{j,n}$ where the $\lambda_{i,j}$'s lie in K_r and the $\varepsilon_{i,j}$'s have p-adic valuation at least $n + c_2$. Moreover, since the action of γ_r is trivial modulo p^{c_2+1}, we deduce $v_p(\lambda_{i,j}) \geq c_2 + 1$. Let $R_r : L \to K_r$ be the Tate's normalized trace defined in Remark 2.1.4. By Proposition 2.1.6 (*cf* also Remark 2.1.7), for all i and j, there exists $\mu_{i,j} \in L$ with $v_p(\mu_{i,j}) \geq n$ and $\varepsilon_{i,j} = R_r(\varepsilon_{i,j}) + \gamma_r\mu_{i,j} - \mu_{i,j}$. For all i, define $v_{i,n+1} = v_{i,n} - \sum_{j=1}^{d} \mu_{i,j}v_{j,n}$. Since the $\mu_{i,j}$'s have all valuation at least n, the items (i) and (ii) are fulfilled. Besides, a simple computation gives:

$$\gamma_r v_{i,n+1} = v_{i,n+1} + \sum_{j=1}^{d} \big(\lambda_{i,j} + R_r(\varepsilon_{i,j})\big) \cdot v_{j,n} + \sum_{j=1}^{d} \gamma_r\mu_{i,j} \cdot \big(v_{j,n} - \gamma_r v_{j,n}\big).$$

Since γ_r acts trivially modulo p^{c_2+1}, the last summand lies in $p^{n+c_2+1}\mathcal{O}_W$. Noting in addition that $v_{j,n} \equiv v_{j,n+1} \pmod{p^n\mathcal{O}_W}$ and that the $\lambda_{i,j}$'s are all divisible by p^{c_2+1}, we obtain the congruence:

$$\gamma_r v_{i,n+1} \equiv v_{i,n+1} + \sum_{j=1}^{d} \big(\lambda_{i,j} + R_r(\varepsilon_{i,j})\big) \cdot v_{j,n+1} \pmod{p^{n+c_2+1}\mathcal{O}_W}$$

from which the item (iii) follows.

Passing to the limit, we obtain an L-basis $v_1, \ldots, v_d$ of W whose K_r-span is stable under the action of $\mathrm{Gal}(K_r/K)$. It remains to prove that it is stable under the whole action of Γ. Let M_0 and M_r be the matrices that gives the action of γ_0 and γ_r on L respectively, that are:

$$\begin{aligned} (\gamma_0 v_1 \quad \cdots \quad \gamma_0 v_d) &= (v_1 \quad \cdots \quad v_d) \cdot M_0 \\ (\gamma_r v_1 \quad \cdots \quad \gamma_r v_d) &= (v_1 \quad \cdots \quad v_d) \cdot M_r. \end{aligned}$$

We do know that M_r has all its entries in K_r and we want to prove that the same holds for M_0. Actually, from our construction of the v_i's, we know further that M_r has integral entries and that it is congruent to the identity matrix modulo p^{c_2+1}. From the commutation of γ_0 and γ_r, we derive the relation $M_0 \cdot \gamma_0 M_r = M_r \cdot \gamma_r M_0$. Define $C = R_r(M_r) - M_r$ where R_r is the Tate's normalized trace. We want to prove that C vanishes.

Since R_r commutes with γ_r, we have the relation $C \cdot \gamma_0 M_r = M_r \cdot \gamma_r C$, from which we derive $\gamma_r C - C = M_r^{-1} \cdot C \cdot \gamma_0 M_r$. Set $N = M_r^{-1} \cdot C \cdot \gamma_0 M_r$ and let v be the smallest valuation of an entry of C. We assume by contradiction that v is finite. The fact that $M_r \equiv I_d \pmod{p^{c_2+1}}$ implies that N is divisible by p^{v+c_2+1}. By unicity in Proposition 2.1.6 (and Remark 2.1.7), we deduce that C must be divisible by p^{v+1}. This contradicts the definition of v. $\square$

Sen's operator. We now put together the results we have established before. Let $W \in \mathrm{Rep}^{\mathrm{f}}_{\mathbb{C}_p}(G_K)$. We define $\hat{W}_\infty = W^{\mathrm{Gal}(\bar{K}/K_\infty)}$ and let W_∞ be the subspace of finite vectors of $\hat{W}_\infty$. Combining Propositions 2.3.2 and 2.3.4, we find that W admits a $\mathbb{C}_p$-basis consisting of elements of W_∞. In other words, the canonical mapping $\mathbb{C}_p \otimes_{K_\infty} W_\infty \to W$ is an isomorphism. The action of G_K on W is then entirely determined by the action of Γ of W_∞. Using the particularly simple structure of Γ, it is possible to describe its action even more concretely.

More precisely, we consider an integer r and a basis $v_1, \ldots, v_d$ of $\hat{W}_\infty$ such that the K_r-vector space $W_r = K_r v_1 \oplus \cdots \oplus K_r v_d$ is stable under the action of Γ (or equivalently, G_K). For $g \in G_K$, we shall denote by $\rho_W(g)$ the endomorphism of W_r given by the action of g. Note that $\rho_W(g)$ is K_r-linear as soon as $g \in \mathrm{Gal}(\bar{K}/K_r)$. In particular $\rho_W(\gamma_s)$ is linear whenever $s \geq r$. Since γ_s converges to the identity in Γ, the logarithm of $\rho_W(\gamma_s)$ is well defined for s sufficiently large. Moreover, we have the relation $p \cdot \log \rho_W(\gamma_{s+1}) = \log \rho_W(\gamma_s)$ as soon as the logarithm $\rho_W(\gamma_s)$ is defined. The sequence $p^{-s} \log \rho_W(\gamma_s)$ is then ultimately constant. Sen's operator Φ_W is defined as the limit of this sequence:

$$\Phi_W = \lim_{s \to \infty} \frac{\log \rho_W(\gamma_s)}{p^s}.$$

We extend Φ_W to W_∞ by K_∞-linearity. This extension is canonical in the sense that it does not depend on the choice of r. Besides, the exponential map can be used to reconstruct the representation W we started with, at least on a finite index subgroup of G. Precisely, there exists an integer s such that:

$$\rho(g) = \exp\big(\alpha(g)\Phi_W\big) \quad \text{for all } g \in \mathrm{Gal}(\bar{K}/K_s) \tag{20}$$

where we recall that $\alpha : G_K \to \mathbb{Z}_p$ was the character defining the isomorphism between $\mathrm{Gal}(K_\infty/K)$ and $\mathbb{Z}_p$. From (20), it follows that the action of $\mathrm{Gal}(\bar{K}/K_s)$ on W can be entirely reconstructed by extending the $\rho(g)$'s to W using semi-linearity.

Example 2.3.6. Consider the representation V given by:

$$G_K \to \mathrm{GL}_2(\mathbb{Q}_p), \quad g \mapsto \begin{pmatrix} 1 & \alpha(g) \\ 0 & 1 \end{pmatrix}$$

already discussed in Example 2.2.11. Set $W = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$. It is easily checked that $W^{\mathrm{Gal}(\bar{K}/K_\infty)} = \hat{K}^2_\infty$ and its subspace of finite vectors is K^2_∞. Sen's operator Φ_V is the nilpotent linear morphism $(x, y) \mapsto (y, 0)$.

Sen's operator exhibits very interesting properties. Below, we state the most important ones.

Proposition 2.3.7. *We keep the above notations. Sen's operator Φ_W is defined over K, in the sense that W_∞ admits a basis in which the matrix of Φ_W has coefficients in K.*

Proof. This follows from the fact that Φ_W commutes with the action of Γ. ☐

Let $\mathrm{Sen}(K, K_\infty)$ denote the category of finite dimensional K_∞-vector spaces equipped with an endomorphism defined over K (in the sense of Proposition 2.3.7). The construction $W \mapsto (W_\infty, \Phi_W)$ defines a functor $\mathcal{S} : \mathrm{Rep}^{\mathrm{f}}_{\mathbb{C}_p}(G_K) \to \mathrm{Sen}(K, K_\infty)$. Indeed a morphism $f : W \to W'$ in the category $\mathrm{Rep}^{\mathrm{f}}_{\mathbb{C}_p}(G_K)$ necessarily maps W_∞ to W'_∞ and commutes with Sen's operators on both sides because it commutes with the Galois action. The functor $\mathcal{S}$ commutes with direct sums, while its behavior under tensor products is governed by the Leibniz rule:

$$(W \otimes W')_\infty = W_\infty \otimes W'_\infty$$
$$\Phi_{W \otimes W'} = \Phi_W \otimes \mathrm{id}_{W'} + \mathrm{id}_W \otimes \Phi_{W'}.$$

Moreover, the functor $\mathcal{S}$ is faithful. Indeed, assume that we are given $W, W' \in \mathrm{Rep}^{\mathrm{f}}_{\mathbb{C}_p}(G_K)$, together with a morphism $f : W \to W'$ such that $\mathcal{S}(f) = 0$. Then, by assumption, f vanishes on the subspace W_∞. Since the latter generates W as a $\mathbb{C}_p$-vector spaces, one must have $f = 0$. In general, $\mathcal{S}$ is not full. However, it detects isomorphisms as shown by the next proposition.

Proposition 2.3.8. *Let $W, W' \in \mathrm{Rep}^{\mathrm{f}}_{\mathbb{C}_p}(G_K)$. We assume that $\mathcal{S}(W)$ and $\mathcal{S}(W')$ are isomorphic in* $\mathrm{Sen}(K, K_\infty)$. *Then W and W' are isomorphic in $\mathrm{Rep}^{\mathrm{f}}_{\mathbb{C}_p}(G_K)$.*

Proof. Let $f : W_\infty \to W'_\infty$ be an isomorphism commuting with Sen's operators. By $\mathbb{C}_p$-linearity, f extends to an isomorphism of $\mathbb{C}_p$-vector spaces $f : W \to W'$. Moreover, thanks to formula (20), there exists an integer s such that f is $\mathrm{Gal}(\bar{K}/K_s)$-equivariant.

Let V be the space of $\mathrm{Gal}(\bar{K}/K_s)$-equivariant $\mathbb{C}_p$-linear morphisms from W to W'. It is endowed with a canonical action of $\mathrm{Gal}(K_s/K)$ and thus appears as an object in the category $\mathrm{Rep}^{\mathrm{f}}_{K_s}(\mathrm{Gal}(K_s/K))$. By Hilbert's theorem 90 (*cf* Theorem 1.3.3), V admits a basis $(f_1, \ldots, f_m)$ of fixed vectors. In other words the f_i's are G_K-equivariant morphisms $W \to W'$. It remains to prove that a suitable K-linear combination of the f_i's is invertible. For this, we consider the m-variate polynomial defined by:

$$P(t_1, t_2, \ldots, t_m) = \det(t_1 f_1 + t_2 f_2 + \cdots + t_m f_m).$$

We know that P is not the zero polynomial because the K_s-span of the f_i's contains an isomorphism (namely f). Since K is an infinite field, P cannot vanish everywhere on K^m. Hence there exist $t_1, \ldots, t_m \in K$ such that $t_1 f_1 + \cdots + t_m f_m$ is an isomorphism. □

Corollary 2.3.9. *A representation $W \in \mathrm{Rep}^{\mathrm{f}}_{\mathbb{C}_p}(G_K)$ is trivial if and only if Sen's operation Φ_W vanishes.*

Proof. It suffices to apply Proposition 2.3.8 with $W' = \mathbb{C}_p^{\dim W}$. □

We conclude our exposition of Sen's theory by noticing that Sen's operator is closely related to the notion of Hodge–Tate representations. Precisely, a representation $V \in \mathrm{Rep}^{\mathrm{f}}_{\mathbb{Q}_p}(G_K)$ is Hodge–Tate if and only if the operator $\Phi_{\mathbb{C}_p \otimes_{\mathbb{Q}_p} V}$ is semi-stable with eigenvalues in $\mathbb{Z}$, these eigenvalues being the Hodge–Tate weights of V. (Combine Examples 2.2.11 and 2.3.6 for an illustration of this property.) Given a general $W \in \mathrm{Rep}^{\mathrm{f}}_{\mathbb{C}_p}(G_K)$, the eigenvalues of Φ_W are sometimes called the *generalized Hodge–Tate weights* of W.

3 Two refined period rings: B_{crys} and B_{dR}

Previously, we have studied the period rings $\mathbb{C}_p$ and B_{HT} and discussed the attached notion of Hodge–Tate representations. In the present section, we introduce two new period rings, called B_{crys} and B_{dR}. As we shall see in §4, these rings have a deeper arithmetical and geometrical content that $\mathbb{C}_p$ and B_{HT}.

The definition of B_{crys} and B_{dR} is a bit elaborated and occupies all this section. In order to ease the task of the reader, we devote two short paragraphs below to collect the most important properties of B_{crys} and B_{dR} and sketch the main steps of their construction.

Before this, we need to recall and introduce some notations. Throughout this section, K will continue to refer to a finite extension of $\mathbb{Q}_p$. Its ring of integers (resp. its residuel field) is denoted by $\mathcal{O}_K$ (resp. k). We define $K_0 = W(k)[\frac{1}{p}]$; it is the maximal unramified extension of $\mathbb{Q}_p$ included in K. We fix an algebraic closure $\bar{K}$ of K and set $G_K = \mathrm{Gal}(\bar{K}/K)$. Observe that $\bar{K}$ is also an algebraic closure of $\mathbb{Q}_p$ and hence does not depend on K. We let K_0^{ur} (resp. K^{ur}) be the maximal unramified extension of K_0 (resp. of K) inside $\bar{K}$. Since K_0 is unramified over $\mathbb{Q}_p$, K_0^{ur} is also the maximal extension of $\mathbb{Q}_p$ inside $\bar{K}$ and thus is also independent of K. We let also $\mathbb{C}_p$ denote the p-adic completion of $\bar{K}$. Finally, we choose and fix once for all a uniformizer π of K.

Main properties of B_{crys} and B_{dR}. As discussed in §1.2, the original idea behind the definition of B_{crys} is the wish to design a variant of Barsotti-type spaces (the $\mathcal{B}$ of §1.2) which includes the tannakian formalism. On the geometric side, a nice tannakian framework in which p-divisible groups naturally arise is crystalline cohomology. Indeed, in many contexts, crystalline cohomology provides powerful invariants that can be used to classify p-divisible groups (and more generally finite flat group schemes) [7]. We then expect the ring B_{crys} to have some "crystalline nature" and to be eventually related to crystalline cohomology. Apart from this, recall that another motivation of p-adic Hodge theory is to compare étale cohomology with de Rham cohomology. The period ring making the comparison possible—namely B_{dR}—then needs to be deeply related to de Rham cohomology. The algebraic structure of B_{crys} and B_{dR} is guided by the above general expectations: the ring B_{crys} (resp. B_{dR}) will carry, as much as possible, the same structures and exhibit similar properties as the crystalline (resp. de Rham) cohomology.

Below, we list the main features of B_{crys} and B_{dR} and, when it is possible, we make the parallel with the corresponding properties of the cohomology. We start with B_{dR}:

- B_{dR} is a discrete valuation field with residue field $\mathbb{C}_p$;
- B_{dR} is an algebra over $\bar{K}\cdot\hat{K}^{\text{ur}}$, but *not* over $\mathbb{C}_p$ (with a defining morphism preserving the Galois action);
- B_{dR} is equipped with a filtration $\text{Fil}^m B_{\text{dR}}$ (which is nothing but the canonical filtration given by the valuation); this filtration corresponds to the de Rham filtration on the cohomology;
- B_{dR} has a distinguished element t on which Galois acts by multiplication by the cyclotomic character; moreover t is a uniformizer of B_{dR}, so that $\text{Fil}^m B_{\text{dR}} = t^m B_{\text{dR}}$;
- the graded ring of B_{dR} is $B_{\text{HT}} = \mathbb{C}_p[t, t^{-1}]$;
- $(B_{\text{dR}})^{G_K} = K$; this property corresponds to the fact that the de Rham cohomology is a vector space over K.

And now for B_{crys}:

- B_{crys} is an algebra over $\hat{K}_0^{\text{ur}}$;
- B_{crys} is equipped with a Frobenius, which is a ring homomorphism $\varphi : B_{\text{crys}} \to B_{\text{crys}}$; this structure corresponds to the action of the Frobenius on the crystalline cohomology;
- there is a canonical embedding $K \otimes_{K_0} B_{\text{crys}} \hookrightarrow B_{\text{dR}}$; this property corresponds to the fact that the crystalline cohomology defines a canonical K_0-structure inside the de Rham cohomology (this is Hyodo–Kato isomorphism);
- the distinguished element t of B_{dR} is in B_{crys};
- $(B_{\text{crys}})^{G_K} = K_0$; this property corresponds to the fact that the de Rham cohomology is a vector space over K_0;
- $\big(B_{\text{crys}} \cap \text{Fil}^0 B_{\text{dR}}\big)^{\varphi=1} = \mathbb{Q}_p$ (the notation "φ=1" means that we are taking the fixed points under the Frobenius).

Sketch of the construction. The starting point of the construction of B_{crys} and B_{dR} is the introduction of the rings A_{inf} and $B^+_{\text{inf}} = A_{\text{inf}}[1/p]$. One may think of A_{inf} as the universal thickening of $\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$; it is obtained *via* a general process (detailed in §3.1) involving a perfectization mecanism as a first step and Witt vectors as a second step. Beyond this purely algebraic construction, it is important to notice that the ring B^+_{inf} has a strong geometrical interpretation. Indeed as observed first by Colmez and then by Fargues–Fontaine [17] and Scholze [38], B^+_{inf} appears at a mixed characteristic analogue[6] of the ring of bounded analytic functions on the open unit disc. Moreover B^+_{inf} is equipped with a Frobenius (coming from the

[6]This analogy has been placed in the framework of Huber geometry by Scholze in [38] and then takes a very substantial meaning. However, for this article, it will be sufficient to keep in mind that elements of B^+_{inf} behave like analytic functions over a nonarchimedian base.

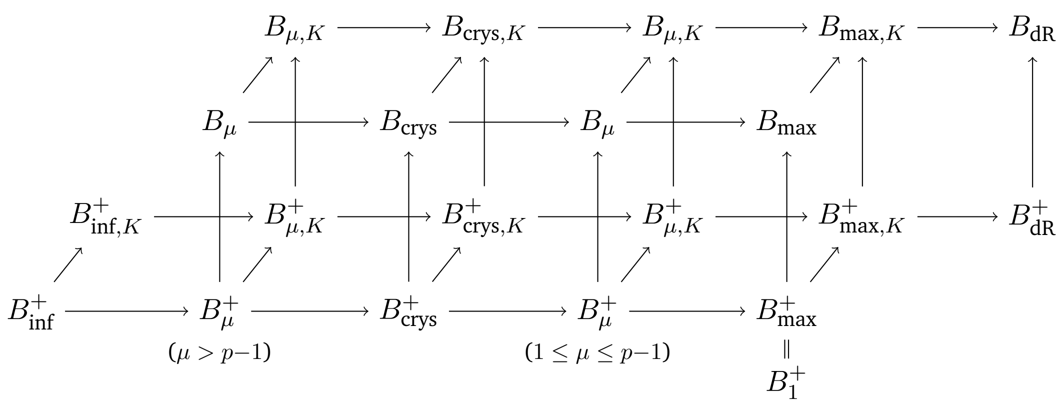

Figure 2: Diagram of period rings; all arrows are injective

general theory of Witt vectors) and a distinguished geometric point, which is materalized by a surjective ring homomorphism $\theta : B^+_{\text{inf}} \to \mathbb{C}_p$.

Following the crystalline formalism, we then define the ring B^+_{crys} as the completion of the divided powers envelope of B^+_{inf} with respect to the ideal $\ker \theta$. Unfortunately, B^+_{crys} does not have a nice geometrical interpretation, in the sense that it is not the ring of analytic functions on a smaller domain. In order to tackle this difficulty, we introduce (following Colmez) some variants of B^+_{crys}. Precisely, given a real parameter $\mu \geq 1$, one considers the rings B^+_μ's of analytic functions defined over some annulus D_μ included in the open unit disc, and containing the distinguished point θ. The B^+_μ's are closely related to B^+_{crys} (we have inclusions in both directions), so that it is often safe to replace the latter by the formers.

Another important feature of B^+_{crys} and the B^+_μ's is that they contain a period of the cyclotomic character, that is an element t on which G_K acts by multiplication by the cyclotomic character. Geometrically, the divisor of t is the orbit of the point θ under the action of the Frobenius, that is the union of all point $\theta \circ \varphi^n$ for n varying in $\mathbb{Z}$. The presence of t in B^+_μ will eventually ensure the admissibility of the representation $\mathbb{Q}_p(\chi_{\text{cycl}}^{-1})$. In order to make $\mathbb{Q}_p(\chi_{\text{cycl}})$ admissible as well (which is of course something we really want to have), we need t to be a unit. So we finally define $B_\mu = B^+_\mu[\frac{1}{t}]$ and the construction of B_{crys} is now complete.

As for the field B_{dR}, it is defined as the fraction field of the completion of the local field of B^+_{inf} (or equivalently, B^+_μ) at the special point θ. The filtration on B_{dR} is nothing but the canonical filtration given by the order of the zero (or the pole) at θ.

The diagram presented on Figure 2 summarizes the period rings we will define in this section and the relations between them. We see on this diagram that the B^+_μ's and the B_μ's all have a variant denoted by an extra index K. They are defined by extending scalars from K_0 to K. These variants are interesting because, when V is a B_μ-admissible representation, the de Rham filtration becomes visible after extending scalars to $B_{\mu,K}$, which is much smaller and sometimes more tractable that B_{dR}.

3.1 Preliminaries: the ring B^+_{inf}

In this subsection, we introduce the ring B^+_{inf} which serves as a common base upon which all the forthcoming constructions will be built.

3.1.1 Perfectization

Let φ denote the Frobenius morphism $x \mapsto x^p$ acting on the quotient $\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p} \simeq \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$ and observe that φ is a ring homomorphism since $\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$ is annihilated by p.

Let $\mathcal{R}$ be the limit of the projective system[7]:

$$\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p} \xrightarrow{\varphi} \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p} \xrightarrow{\varphi} \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p} \xrightarrow{\varphi} \cdots \xrightarrow{\varphi} \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p} \xrightarrow{\varphi} \cdots$$

Concretely, an element of $\mathcal{R}$ is a sequence $(\xi_n)_{n\geq 0}$ of elements of $\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$ satisfying the following compatibility property: $\xi_{n+1}^p = \xi_n$ for all $n \geq 0$. Clearly, $\mathcal{R}$ is a ring of characteristic p.

In a slight abuse of notation, we continue to write φ for the Frobenius acting on $\mathcal{R}$. Over this ring, it is an isomorphism, its inverse being given by the shift map $(\xi_0, \xi_1, \xi_2, \ldots) \mapsto (\xi_1, \xi_2, \xi_3, \ldots)$. We sometimes say that $\mathcal{R}$ is the *perfectization* of $\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$. Moreover $\mathcal{R}$ is endowed with an action of G_K coming from its natural action on $\mathcal{O}_{\mathbb{C}_p}$.

Some distinguished elements. Choose a primivite p-root of unity in $\mathcal{O}_{\bar{K}}$ and denote it by ε_1. Similarly, choose a p-th root of ε_1 and denote it by ε_2; obviously, ε_2 is a primitive p^2-th root of unity. Repeating inductively this process, we construct elements $\varepsilon_3, \varepsilon_4, \ldots \in \mathcal{O}_{\bar{K}}$ such that $\varepsilon_{n+1}^p = \varepsilon_n$ for all n. Let $\bar{\varepsilon}_n \in \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$ be the image of ε_n. The compabitility property ensures that the sequence $(1, \bar{\varepsilon}_1, \bar{\varepsilon}_2, \ldots)$ defines an element in $\mathcal{R}$; we shall denote it by $\underline{\varepsilon}$. We emphasize that $\underline{\varepsilon}$ does depend on the choice of the ε_n's. However, the dependency is easy to write down explicitely: if $(\varepsilon'_n)_{n\geq 0}$ is another compatible sequence of primitive p^n-th roots of unity, one can always find an element $g \in G_K$ such that $\varepsilon'_n = g\varepsilon_n = \varepsilon_n^{\chi_{\mathrm{cycl}}(g)}$. Hence the element of $\mathcal{R}$ defined the ε'_n's is $\underline{\varepsilon}^{\chi_{\mathrm{cycl}}(g)}$. In what follows, we fix once for all an element $\underline{\varepsilon}$ as above.

In a similar fashion, we choose a compatible system $(p_n)_{n\geq 1}$ of p^n-root of p, *i.e.* $p_1^p = p$ and $p_{n+1}^p = p_n$ for all $n \geq 1$. If $\bar{p}_n \in \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$ is the reduction of p_n modulo p, the sequence $(0, \bar{p}_1, \bar{p}_2, \ldots)$ defines an element of $\mathcal{R}$ that we will denote by $p^\flat$. Again, $p^\flat$ depends on the choice of the p_n's but we can check that another choice would finally lead to an element of the form $p^\flat \cdot \underline{\varepsilon}^a$ for some $a \in \mathbb{Z}_p$. The same construction works more generally if we start for any element $x \in \mathcal{O}_{\mathbb{C}_p}$ in place of p; it leads to an element $x^\flat \in \mathcal{R}$, which is well defined up to multiplication by $\underline{\varepsilon}^a$ with $a \in \mathbb{Z}_p$. Besides $p^\flat$, we will fix a choice of $\pi^\flat$ (where we recall that π is a fixed uniformizer of K) for future use.

Valuation. The ring $\mathcal{R}$ is equipped with a derivation $v_\flat$ that we are going to define now. We start with the following observation: if x is a nonzero element in $\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$, the p-adic valuation of $\hat{x}$ does not depend on the lifting $\hat{x}$ of x. The valuation v_p then induces a well defined function $v_p : \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p} \to \mathbb{Q} \cup \{+\infty\}$ where we agree that $v_p(0) = +\infty$ as usual. For $\xi = (\xi_n)_{n\geq 0}$ in $\mathcal{R}$, we define:

$$v_\flat(\xi) = \lim_{n\to\infty} p^n v_p(\xi_n).$$

The compatibility condition $\xi_{n+1}^p = \xi_n$ implies that the sequence $\big(p^n v_p(\xi_n)\big)_{n\geq 0}$ is ultimately constant; so the limit is well defined. The function $v_\flat$ satisfies the following properties for $\xi, \xi' \in \mathcal{R}$:

(1) $v_\flat(\mathcal{R}) = \mathbb{Q} \cup \{+\infty\}$,

(2) $v_\flat(\xi) = \infty$ if and only if $\xi = 0$,

(3) $v_\flat(\xi) = 0$ if and only if ξ is invertible,

(4) $v_\flat(\xi + \xi') \geq \min(v_\flat(\xi), v_\flat(\xi'))$ and equality holds if $v_\flat(\xi) \neq v_\flat(\xi')$,

(5) $v_\flat(\xi\xi') = v_\flat(\xi) + v_\flat(\xi')$.

[7]This definition is s special case of a general construction (the *tilt*) in the theory of perfectoid spaces [36]. We refer to Andreatta and al. lecture [1] in this volume for an introduction to perfectoid spaces. The notations $p^\flat$, $\pi^\flat$ and $v_\flat$ that we will introduce later comes from the language of perfectoid spaces.

Combining (2) and (5), we find that $\mathcal{R}$ is a domain. Indeed if ξ and ξ' are nonzero elements of $\mathcal{R}$, then $v_\flat(\xi)$ and $v_\flat(\xi')$ are finite, and so $v_\flat(\xi\xi') = v_\flat(\xi) + v_\flat(\xi')$ is also finite. The existence of $v_\flat$ implies that $\mathcal{R}$ is a local ring with maximal ideal $\mathfrak{m}_\mathcal{R}$ consisting of elements of positive valuation. The residue field $\mathcal{R}/\mathfrak{m}_\mathcal{R}$ is canonically isomorphic to $\bar{k}$. We observe in addition that the projection $\mathcal{R} \to \bar{k}$ has a canonical splitting defined by:

$$a \mapsto \big([a] \bmod p, [a^{1/p}] \bmod p, [a^{1/p^2}] \bmod p, \dots\big)$$

where the notation $[\cdot]$ stands for the Teichmuller representative. Besides, the valuation $v_\flat$ equips $\mathcal{R}$ with a distance, and hence a topology. The Galois action on $\mathcal{R}$ preserves $v_\flat$; in particular, it is continuous.

An easy consequence of the existence of a valuation is the following result.

Lemma 3.1.1. *The projection onto the first coordinate $\mathcal{R} \to \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$ induces an isomorphism $\mathcal{R}/p^\flat\mathcal{R} \simeq \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$.*

Proof. Let $f : \mathcal{R} \to \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$, $(\xi_0, \xi_1, \dots) \mapsto \xi_0$. The surjectivity of f is a consequence of the fact that $\mathbb{C}_p$ is algebraically closed. On the other hand, it is clear that the kernel of f consists of elements ξ such that $v_\flat(\xi) \geq 1$. Since $v_\flat(p^\flat) = 1$, we deduce that $\ker f$ is the principal ideal generated by $p^\flat$. This proves the lemma. □

3.1.2 Witt vectors

We set $A_{\text{inf}} = W(\mathcal{R})$ (where $W(-)$ stands for the Witt vectors functor) and $B^+_{\text{inf}} = A_{\text{inf}}[\frac{1}{p}]$. For $x \in \mathcal{R}$, we let $[x]$ denote its representative Teichmüller in A_{inf}. Since $\mathcal{R}$ is perfect, an element of A_{inf} can be written uniquely as a convergent series $\sum_{i\geq 0}[\xi_i]\, p^i$ with $\xi_i \in \mathcal{R}$ for all i. A similar decomposition holds for elements in B^+_{inf}: each such element x has a unique expansion of the form $\sum_{i\geq i_0}[\xi_i]\, p^i$ (with $\xi_i \in \mathcal{R}$) where i_0 is a (possibly negative) integer, which depends on x.

The inclusion $\bar{k} \to \mathcal{R}$ provides by functoriality a ring morphism $W(\bar{k}) \to A_{\text{inf}}$. Thus $\hat{K}_0^{\text{ur}}$ embeds into B^+_{inf}. The ring A_{inf} is a local ring whose maximal ideal is the kernel of the composition $A_{\text{inf}} \to W(\bar{k}) \to \bar{k}$ where the first map is induced by the projection $\mathcal{R} \to \bar{k}$ and the second map is the reduction modulo p. Concretely, it consists of series $\sum_{i\geq 0}[\xi_i]\, p^i$ for which $\xi_0 \in \mathfrak{m}_\mathcal{R}$.

We set $A_{\text{inf},K} = \mathcal{O}_K \otimes_{W(k)} A_{\text{inf}}$ and $B^+_{\text{inf},K} = K \otimes_{K_0} B^+_{\text{inf}}$. These tensors products make sense because we saw that A_{inf} is an algebra over $W(k)$. The elements of $B^+_{\text{inf},K}$ have a canonical expansion of the form $\sum_{i\geq i_0}[\xi_i]\, \pi^i$ with $i_0 \in \mathbb{Z}$ and $\xi_i \in \mathcal{R}$ for all $i \geq i_0$. Moreover $A_{\text{inf},K}$ is a local ring and its maximal ideal consists of series as above such that $\xi_0 \in \mathfrak{m}_\mathcal{R}$; its residue field is $\bar{k}$.

Additional structures. By definition of the Witt vectors, B^+_{inf} carries an action of a Frobenius, that we shall continue to call φ. On the above representation, it is given by the simple formula:

$$\varphi\left(\sum_{i=i_0}^{\infty}[\xi_i]\, p^i\right) = \sum_{i=i_0}^{\infty}[\xi_i^p]\, p^i \qquad (i_0 \in \mathbb{Z},\ \xi_i \in \mathcal{R}). \tag{21}$$

We emphasize that φ does not admit a *canonical* extension to $B^+_{\text{inf},K}$ as there is no canonical Frobenius on K.

The ring B^+_{inf} is also equipped with an action of G_K by functorialiy of Witt vectors. Again, this action has a simple expression, namely:

$$g\left(\sum_{i=i_0}^{\infty}[\xi_i]\, p^i\right) = \sum_{i=i_0}^{\infty}[g\xi_i]\, p^i \qquad (i_0 \in \mathbb{Z},\ \xi_i \in \mathcal{R}) \tag{22}$$

for all $g \in G_K$. The G_K-action extends to $B^+_{\text{inf},K}$ by letting G_K act trivially on $\mathcal{O}_K$.

Finally, we equip A_{inf} and $A_{\text{inf},K}$ with the *weak topology*, which is the topology defined by the ideal $(p, [p^\flat])$ (or equivalently, by the ideal $(p, [x])$ for any element $x \in \mathfrak{m}_{\mathcal{R}}$). Concretely, if

$$x_n = \sum_{i=0}^{\infty} [\xi_{i,n}]\, p^i \in A_{\text{inf}} \quad \text{and} \quad x = \sum_{i=0}^{\infty} [\xi_i]\, p^i \in A_{\text{inf}}$$

the sequence $(x_n)_{n \geq 0}$ converges to x if $\xi_{i,n} \to \xi_i$ for each fixed index $i \in \mathbb{N}$, and a similar property holds for $A_{\text{inf},K}$. The topology on A_{inf} induces a topology on the subset $p^{-v} A_{\text{inf}}$ of B_{inf}^+ for all v. Gluing them, we obtain a topology on $B_{\text{inf}}^+ = \bigcup_{v \geq 0} p^{-v} A_{\text{inf}}$. In concrete terms, a sequence $(x_n)_{n \geq 0}$ of elements on B_{inf}^+ converges to $x \in B_{\text{inf}}^+$ if and only if there exists an integer v such that $p^v x_n \in A_{\text{inf}}$ for all n and $p^v x_n$ tends to $p^v x$ in A_{inf} as n goes to infinity. The topology on $B_{K,\text{inf}}^+$ is defined similarly.

From the above descriptions, it follows that the Frobenius acts continuously on A_{inf} and G_K acts continuously on A_{inf} and $A_{\text{inf},K}$.

Newton polygons. In [17], Fargues and Fontaine argue that elements of A_{inf} (resp. $A_{\text{inf},K}$) should be thought of as analytic functions of the variable p (resp. π); indeed, they share many properties with bounded analytic functions on the open unit disc. Similarly, elements of B_{inf}^+ and $B_{\text{inf},K}^+$ resemble to bounded analytic functions on the punctured open unit disc.

In particuler, there is a well-defined notion of Newton polygons for series in B_{inf}^+ and $B_{\text{inf},K}^+$. Precisely, if $x = \sum_{i \geq i_0} [\xi_i]\, p^i \in A_{\text{inf}}$, its *Newton polygon* is defined as the convex hull in $\mathbb{R}^2$ of the points $(i, v_\flat(\xi_i))$ together with two points at infinity in the direction of the positive x-axis and the direction of the positive y-axis respectively. Similarly, the Newton polygon of $x = \sum_{i \geq i_0} [\xi_i]\, \pi^i \in A_{\text{inf},K}$ is the convex hull of the points $(\frac{i}{e}, v_\flat(\xi_i))$ and the same points at infinity. Using that $\pi^e = up$ for some invertible element $u \in \mathcal{O}_K$, one easily proves that the above definition coincides with that of Newton polygons on B_{inf}^+ when x is in B_{inf}^+. Let $\text{NP}_{\text{inf}}(x)$ denote the Newton polygon of $x \in B_{\text{inf},K}^+$.

Fargues and Fontaine prove that Newton polygons satisfy many excepted properties. For example, they are multiplicative in the sense that $\text{NP}_{\text{inf}}(xy) = \text{NP}_{\text{inf}}(x) + \text{NP}_{\text{inf}}(y)$ where the plus sign on the right hand side denotes the Minkowski sum. Moreover Fargues and Fontaine prove an analogue of the Weierstrass preparation and factorization theorems in this context, showing that Newton polygons serve as a guide for factorization in the rings B_{inf}^+ and $B_{\text{inf},K}^+$ as they do for usual analytic functions. We do not reproduce their proofs here because we will only use Newton polygons for visualizing our forthcoming constructions, and not for proving results. In any case, we refer to [17, §1–3] for many developments in this direction.

We conclude this discussion by examining the action of the additional structures at the level of Newton polygons. Since G_K acts on $\mathcal{R}$ by isometries, it follows from the formula (22) that $\text{NP}_{\text{inf}}(gx) = \text{NP}_{\text{inf}}(x)$ whenever g is in G_K and x is in $B_{\text{inf},K}^+$. As for Frobenius, formula (21) shows that, for any $x \in B_{\text{inf}}^+$, we have $\text{NP}_{\text{inf}}(\varphi(x)) = \varphi_{\mathbb{R}^2}(\text{NP}_{\text{inf}}(x))$ where $\varphi_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^2$ takes (i, v) to (i, pv).

3.1.3 The "sharp" construction

In §3.1, starting with $x \in \mathcal{O}_{\mathbb{C}_p}$, we have constructed an element $x^\flat \in \mathcal{R}$ (which was only well-defined up to multiplication by an element of the form $\underline{\varepsilon}^a$ with $a \in \mathbb{Z}_p$). Let us recall more precisely that the element $x^\flat = (x_0 \bmod p, x_1 \bmod p, x_2 \bmod p, \ldots)$ where $x_0 = x$ and x_{n+1} is a p-th root of x_n for $n \geq 0$.

It turns out that the datum of $x^\flat$ entirely determines x. Precisely, if we write $x^\flat = (\xi_0, \xi_1, \xi_2, \ldots)$ and if we choose a lifting $\hat{\xi}_n \in \mathcal{O}_{\mathbb{C}_p}$ of x_n for all n, we have $x = \lim_{n \to \infty} \hat{\xi}_n^{p^n}$ independently of the choices of the liftings. Indeed, following the definitions, we find that $x_n \equiv \hat{\xi}_n \pmod{p}$ and then, raising to the p^n-th power, $x \equiv \hat{\xi}_n^{p^n} \pmod{p^{n+1}}$. This motivates the following definition.

Definition 3.1.2. For $\xi = (\xi_0, \xi_1, \xi_2, \dots) \in \mathcal{R}$, we put

$$\xi^\sharp = \lim_{n\to\infty} \hat{\xi}_n^{p^n}$$

where $\hat{\xi}_n$ is a lifting of ξ_n.

One checks immediately that the function $\mathcal{R} \to \mathcal{O}_{\mathbb{C}_p}$, $\xi \mapsto \xi^\sharp$ is surjective and multiplicative. Its "kernel" is the closed subgroup of $\mathcal{R}^\times$ generated by $\underline{\varepsilon}$; it is isomorphic to $\mathbb{Z}_p$. Besides, we observe that $v_p(\xi^\sharp) = v_\flat(\xi)$ for all $\xi \in \mathcal{R}$ and that $\xi^\sharp$ is the Teichmuller representative of ξ if ξ is in $\bar{k}$. By the general properties of Witt vectors, the "sharp" function extends to a surjective homorphism of $\hat{K}_0^{\text{ur}}$-algebras $\theta : B^+_{\text{inf}} \to \mathbb{C}_p$ which commutes with the G_K-action. Concretely, it is given by:

$$\theta : \quad \sum_{i=i_0}^{\infty} [\xi_i]\, p^i \;\mapsto\; \sum_{i=i_0}^{\infty} \xi_i^\sharp\, p^i \qquad (i_0 \in \mathbb{Z},\ \xi_i \in \mathcal{R}).$$

Note that the latter series converges in $\mathbb{C}_p$ since its i-th summand is a multiple of p^i. The morphism θ extends by K-linearity to a surjective G_K-equivariant homomorphism of $\hat{K}^{\text{ur}}$-algebras $\theta_K : B^+_{\text{inf},K} \to \mathbb{C}_p$.

Proposition 3.1.3. *(i) Let $z \in A_{\text{inf}}$ be an element such that $\theta(z) = 0$ and $v_\flat(z \bmod p) = 1$. Then z generates $A_{\text{inf}} \cap \ker \theta$, viewed as an ideal of A_{inf}.*

(ii) Let $z \in A_{\text{inf},K}$ be an element such that $\theta_K(z) = 0$ and $v_\flat(z \bmod \pi) = \frac{1}{e}$. Then z generates $A_{\text{inf},K} \cap \ker \theta_K$, viewed as an ideal of $A_{\text{inf},K}$.

Remark 3.1.4. In particular, an element z satisfying the condition of the first item (resp. the second item) of Proposition 3.1.3 is a generator of the ideal $\ker \theta$ (resp. $\ker \theta_K$).

Proof of Proposition 3.1.3. Let $z \in A_{\text{inf}}$ such that $\theta(z) = 0$ and $v_\flat(\zeta) = 1$ with $\zeta = z \bmod p$. Let $x \in \ker \theta \cap A_{\text{inf}}$. Write $x = \sum_{i\geq 0} [\xi_i]\, p^i$ with $\xi_i \in \mathcal{R}$. From $\theta(x) = 0$, we derive that $v_p(\xi_0^\sharp) \geq 1$ and then $v_\flat(\xi_0) \geq 1$. From the assumption $v_\flat(\zeta) = 1$, we find that ζ divides ξ_0 in $\mathcal{R}$. Thus, we can write $x = zy_0 + px_1$ with $y_0, x_1 \in A_{\text{inf}}$. From this above equality, we derive $\theta(x_1) = 0$ and we can then repeat the argument with x_1, ending up with a writing of the form $x = z{\cdot}(y_0 + py_1) + p^2 x_2$ with $y_1, x_2 \in A_{\text{inf}}$. Repeating this process again and again, we construct a sequence $(y_n)_{n\geq 0}$ of elements of A_{inf} such that:

$$x \equiv z \cdot (y_0 + py_1 + \cdots + p^n y_n) \pmod{p^n A_{\text{inf}}}$$

for all n. Passing to the limit we find that $x \in zA_{\text{inf}}$, which proves (i).

The statement (ii) is proved similarly. □

We remark that there do exist elements in A_{inf} satisfying the condition of Proposition 3.1.3. The simplest one is $[p^\flat] - p$, which is then a generator of $\ker \theta$. Similarly $[\pi^\flat] - \pi \in A_{\text{inf},K}$ satisfies the condition of Proposition 3.1.3 and so is a generator of $\ker \theta_K$. Another generator of $\ker \theta$ is $E([\pi^\flat])$ where E is minimal polynomial of π over K_0. Indeed, on the one hand, we have $\theta(E([\pi^\flat])) = E(\theta([\pi^\flat])) = E(\pi) = 0$ and, on the other hand, $E([\pi^\flat])$ reduces modulo p to the constant coefficient of E, which has valuation 1.

The next proposition gives another quite interesting generator of $\ker \theta$.

Proposition 3.1.5. *The element*

$$\omega = \frac{[\underline{\varepsilon}] - 1}{[\underline{\varepsilon}^{1/p}] - 1} = [\underline{\varepsilon}^{1/p}] + [\underline{\varepsilon}^{1/p}]^2 + \cdots + [\underline{\varepsilon}^{1/p}]^{p-1}$$

satisfies the condition of Proposition 3.1.3.(i).

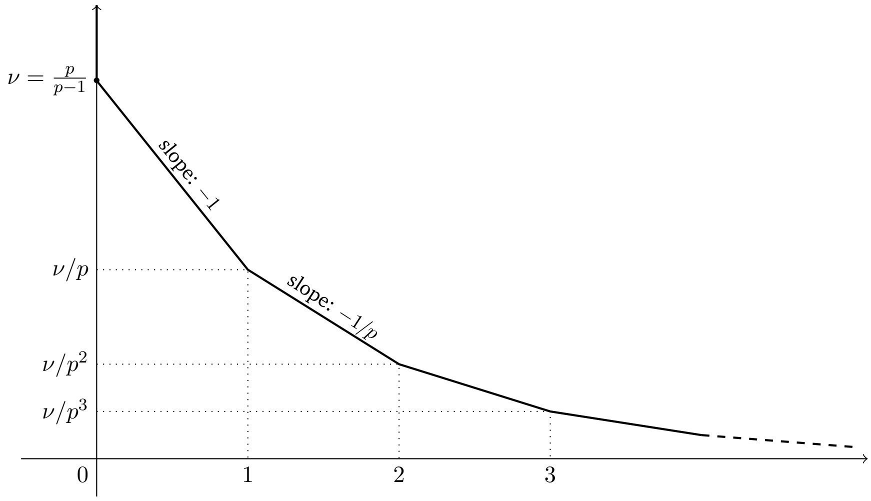

Figure 3: The Newton polygon of $[\underline{\varepsilon}] - 1$

Proof. We want to check that $\theta(\omega) = 0$ and $v_\flat(\omega \bmod p) = 1$. The first equality follows from the fact that $\theta([\underline{\varepsilon}]) = 1$ and the fact that $\theta([\underline{\varepsilon}^{1/p}])$ is a primitive p-th root of unity. Let us now prove that $v_\flat(\omega \bmod p) = 1$. Reducing modulo p, we find that $\omega \bmod p = \frac{\underline{\varepsilon}-1}{\underline{\varepsilon}^{1/p}-1}$. Write $\underline{\varepsilon} = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots)$ where ε_n is the reduction modulo p of a primitive p^n-th root of unity. Coming back to the definition of $v_\flat$, we find:

$$v_\flat(\omega \bmod p) = \lim_{n\to\infty} p^{n+1} \cdot v_p\left(\frac{\varepsilon_n - 1}{\varepsilon_{n+1} - 1}\right). \tag{23}$$

By the standard properties of the cyclotomic extension (*cf* [39, Chap. IV, §4]), we know that the p-adic valuation of $\varepsilon_n - 1$ is $\frac{1}{p^n(p-1)}$. Injecting this in (23), we obtain $v_\flat(\omega \bmod p) = \frac{p}{p-1} - \frac{1}{p-1} = 1$. □

Remark 3.1.6. Since two generators of $\ker \theta$ differ by multiplication by a unit, they have to share the same Newton polygon up to translation by a horizontal vector. If in addition, they satisfy the conditions of Proposition 3.1.3, the Newton polygons must coincide since they both admit $(0, 1)$ as an extremal point. Clearly, the Newton polygon of $[p^\flat] - p$ is the convex polygon whose vertices are $(0, +\infty)$, $(0, 1)$, $(1, 0)$ and $(+\infty, 0)$. The Newton polygon of ω is then the same. Writing

$$[\underline{\varepsilon}] - 1 = \prod_{n=0}^{\infty} \varphi^{-n}(\omega) \tag{24}$$

and using the multiplicative properties of the Newton polygons, we find that $\mathrm{NP}_{\mathrm{inf}}([\underline{\varepsilon}] - 1)$ starts at $(0, \frac{1}{p-1})$ and then has a segment of length 1 of slope p^{-n} for each nonnegative integer n (*cf* Figure 3).

Proposition 3.1.7. *The element $[\underline{\varepsilon}] - 1$ is a generator of the ideal $\bigcap_{n\geq 0} \ker(\theta \circ \varphi^n)$.*

Remark 3.1.8. Proposition 3.1.7 is not surprising after formula (24). Indeed if x is such that $\theta \circ \varphi^n(x) = 0$ for all $n \geq 0$, then x must to divisible by $\varphi^{-n}(\omega)$ for all $n \geq 0$. It is then reasonable to expect to $[\underline{\varepsilon}] - 1 = \prod_{n=0}^{\infty} \varphi^{-n}(x)$ divides x since the Newton polygon of the factors do not share any common slope (and thus the factors look pairwise coprime). It is possible to turn this vague idea into a rigourous proof. However, we prefer giving below a more direct argument, which is easier to write down.

Proof. Clearly $[\underline{\varepsilon}] - 1 \in \bigcap_{n\geq 0} \ker(\theta \circ \varphi^n)$. Repeating the second part of the proof of Proposition 3.1.3, we are reduced to show that any element $x \in A_{\mathrm{inf}}$ such that $\theta \circ \varphi^n(x) = 0$

verifies $v_\flat(x \bmod p) \geq \frac{p}{p-1}$. From $\theta(x) = 0$, we deduce that x can be written ωx_1 with $x_1 \in A_{\text{inf}}$. Since $\theta \circ \varphi(\omega) \neq 0$, we deduce that $\theta \circ \varphi(x_1)$ must vanish. Therefore there exists $x_2 \in A_{\text{inf}}$ such that $x_1 = \varphi^{-1}(\omega)x_2$, *i.e.* $x = \omega\varphi^{-1}(\omega)x_2$. By induction, we find that x has to be divisible by $x = \omega\varphi^{-1}(\omega)\cdots\varphi^{-n}(\omega)$ in A_{inf} for all n. Reducing modulo p, this implies $v_\flat(x \bmod p) \geq 1 + \frac{1}{p} + \cdots + \frac{1}{p^n}$ for all n. Passing to the limit, we find $v_\flat(x \bmod p) \geq \frac{p}{p-1}$ as expected. □

3.2 The ring B_{crys} and some variants

In this subsection, we introduce the ring B_{crys} and its variants B_μ's. The former is interesting because it fits very well in the crystalline framework and therefore is well suited for studying cohomology. Nevertheless, as we shall see, B_{crys} does not behave very well from the purely algebraic point of view. The B_μ's are substitutes to B_{crys} which share its most important features and, in addition, exhibit better algebraic (and analytic) properties, and hence are easier to work with.

3.2.1 Divided powers

Given $x \in A_{\text{inf}}$, we denote by $A_{\text{inf}}\langle x \rangle$ the sub-A_{inf}-algebra of B^+_{inf} generated by the elements $\frac{x^n}{n!}$ for n varying in $\mathbb{N}$. Obviously if y divides x in A_{inf}, we have $A_{\text{inf}}\langle x \rangle \subset A_{\text{inf}}\langle y \rangle$. In particular, $A_{\text{inf}}\langle x \rangle$ only depends on the principal ideal xA_{inf}. By the proof of Proposition 3.1.3, we know that $A_{\text{inf}} \cap \ker\theta$ is a principal ideal of A_{inf}. The following definition then makes sense.

Definition 3.2.1. We define A_{crys} as the p-adic completion of $A_{\text{inf}}\langle z \rangle$ where z is some generator of the ideal $A_{\text{inf}} \cap \ker\theta$. We set $B^+_{\text{crys}} = A_{\text{crys}}[\frac{1}{p}]$.

Rephrasing the definition, we can write:

$$A_{\text{crys}} = A_{\text{inf}}\left\langle [p^\flat] - p \right\rangle^\wedge = A_{\text{inf}}\langle \omega \rangle^\wedge$$

where the exponent "$\wedge$" means the p-adic completion and the element ω is the one of Proposition 3.1.5.

Lemma 3.2.2. *For $x, y \in A_{\text{inf}}$ with $x \equiv y \pmod{pA_{\text{inf}}}$, we have $A_{\text{inf}}\langle x \rangle = A_{\text{inf}}\langle y \rangle$.*

Proof. By symmetry, it is enough to prove that $A_{\text{inf}}\langle x \rangle \subset A_{\text{inf}}\langle y \rangle$, *i.e.* that $\frac{x^n}{n!} \in A_{\text{inf}}\langle y \rangle$ for all positive integer n. Writing $x = y + pz$ with $z \in A_{\text{inf}}$, we have:

$$\frac{x^n}{n!} = \frac{(y+pz)^n}{n!} = \sum_{i=0}^{n} \frac{p^i z^i}{i!} \cdot \frac{y^{n-i}}{(n-i)!}. \tag{25}$$

We recall that $v_p(i!) = \frac{i - s_p(i)}{p-1}$ where $s_p(i)$ denotes the sum of the digits of i in radix p. In particular, we observe that $v_p(i!) \leq i$, so that the fraction $\frac{p^i}{i!}$ is in $\mathbb{Z}_p$. The formula (25) then presents $\frac{x^n}{n!}$ as an A_{inf}-linear combination of elements of the form $\frac{y^j}{j!}$ for j between 0 and n. Therefore, $\frac{x^n}{n!} \in A_{\text{inf}}\langle y \rangle$ and we are done. □

The above lemma shows that $A_{\text{crys}} = A_{\text{inf}}\left\langle [p^\flat] \right\rangle^\wedge$. Since $A_{\text{inf}}\langle x \rangle$ depends only on the ideal generated by x, we also have $A_{\text{crys}} = A_{\text{inf}}\left\langle [\xi] \right\rangle^\wedge$ for any element $\xi \in \mathcal{R}$ with $v_\flat(\xi) = 1$.

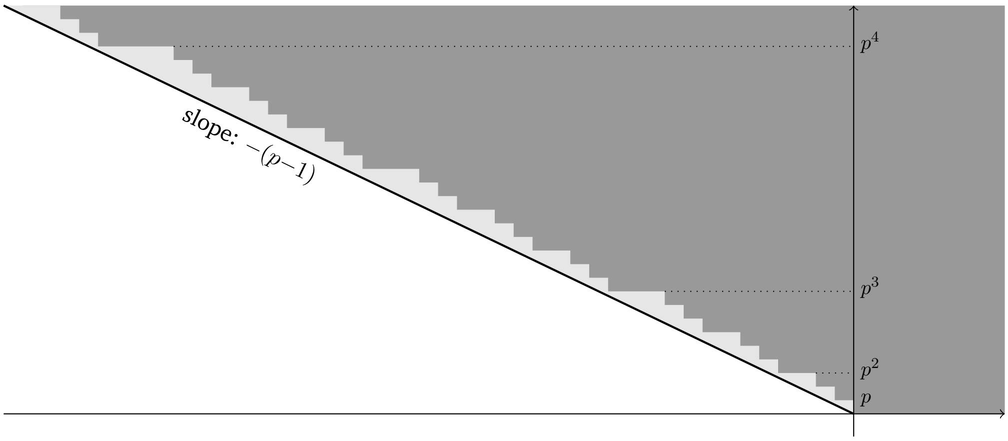

Figure 4: Convergence conditions for elements in A_{crys}

Topology and additional structures. Since A_{crys} is defined as a p-adic completion, it is quite natural to endow A_{crys} (and B^+_{crys}) with the p-adic topology. Noticing that we can obviously write $[p^\flat]^n = n! \cdot \frac{[p^\flat]^n}{n!}$, it follows from the defintion of A_{crys} that $[p^\flat]^n$ tends to zero when n goes to infinity (since $n!$ goes to zero for the p-adic topology). In particular the inclusion $A_{\text{inf}} \to A_{\text{crys}}$ is continuous. Inverting p, we find that the inclusion $B^+_{\text{inf}} \to B^+_{\text{crys}}$ is continuous as well.

Besides, we observe that the Frobenius extends canonically to a ring homomorphism $\varphi : A_{\text{crys}} \to A_{\text{crys}}$. This can be checked by noticing that $A_{\text{inf}}\langle [p^\flat] \rangle$ is stable under the Frobenius since $\varphi\big(\frac{[p^\flat]^n}{n!}\big) = [p^\flat]^{np-n} \cdot \varphi\big(\frac{[p^\flat]^n}{n!}\big)$. Inverting p, we obtain an extension of the Frobenius to B^+_{crys}. We shall continue to denote it by φ in the sequel. Similarly, the action of G_K extends to B^+_{crys}.

The embedding $W(\bar{k}) \to A_{\text{inf}} \to A_{\text{crys}}$ endows A_{crys} with a structure of $W(\bar{k})$-algebra. Similarly B^+_{crys} is an algebra over $\hat{K}_0^{\text{ur}}$. It then makes sense to define $A_{\text{crys},K} = \mathcal{O}_K \otimes_{W(k)} A_{\text{crys}}$ and $B^+_{\text{crys},K} = K \otimes_{K_0} B^+_{\text{crys}} = A_{\text{crys},K}[\frac{1}{p}]$.

3.2.2 Some analytic analogues of A_{crys}

In §3.1, we saw that elements of A_{inf} admitted a nice series expansion, allowing for an analytic interpretation of the ring A_{inf}. To some extent, this point of view is also meaningful for A_{crys}. Indeed, it follows from $A_{\text{crys}} = A_{\text{inf}}\langle [p^\flat] \rangle^\wedge$, that any element $x \in A_{\text{crys}}$ has a unique expansion of the form:

$$\begin{gathered} x = \sum_{i \in \mathbb{Z}} [\xi_i]\, p^i \qquad (\xi_i \in \mathcal{R}) \\ \text{with} \quad v_\flat(\xi_i) - \nu(i) \geq 0 \quad \text{and} \quad \lim_{i \to -\infty} v_\flat(\xi_i) - \nu(i) = +\infty \end{gathered} \tag{26}$$

where, for $i \geq 0$, $\nu(i) = 0$ and, for $i < 0$, $\nu(i)$ denotes the smallest integer n such that $v_p(n!) + i \geq 0$. From the formula $v_p(n!) = \frac{n - s_p(n)}{p-1} = \frac{n}{p-1}$, we derive that, for $i \ll 0$, we have the estimation:

$$-i \cdot (p-1) \leq \nu(i) \leq -i \cdot (p-1) + O\big(\log|i|\big). \tag{27}$$

We insist on the fact that the term $O(\log|i|)$ is not bounded (it may have order of magnitude $(p-1)\cdot\frac{\log|i|}{\log p}$); hence, we cannot replace $\nu(i)$ by $-i\cdot(p-1)$ in (26). We will circumvent this difficulty later on. Figure 4 illustrates the convergence conditions discussed above: the grey part is the region on which $v_\flat(\xi_i) - \nu(i) \geq 0$.

Analytic functions on annuli. The function ν that appeared in the formula (26) has a very erratic behavior. This is unfortunate for two reasons: the ring A_{crys} we defined do not have pleasant algebraic properties (for instance, it is not noetherian), nor a nice analytic interpretation (its elements are not anayltic functions defined on a nice domain). In order to get around these difficulties, we introduce a variant of A_{crys} which does not have these defaults. More precisely, given a positive real number μ, we introduce the ring A_μ consisting of series of the form:

$$x = \sum_{i\in\mathbb{Z}} [\xi_i]\, p^i \qquad (\xi_i \in \mathcal{R})$$
$$\text{with} \quad v_\flat(\xi_i) + \mu i \geq 0 \quad \text{and} \quad \lim_{i\to-\infty} v_\flat(\xi_i) + \mu i = +\infty.$$

When μ is rational[8], A_μ is the p-adic completion of $A_{\text{inf}}\big[\frac{[\xi]}{p}\big]$ for any $\xi \in A_{\text{inf}}$ with $v_\flat(\xi) = \mu$. We let $B_\mu^+ = A_\mu[\frac{1}{p}]$. The elements of B_μ^+ are series of the form:

$$x = \sum_{i\in\mathbb{Z}} [\xi_i]\, p^i \qquad (\xi_i \in \mathcal{R})$$
$$\text{with} \quad \lim_{i\to-\infty} v_\flat(\xi_i) + \mu i = +\infty$$

i.e. the same conditions as for A_μ except that the condition of positivity has been dropped. From the analytic point of view, elements of B_μ^+ should be considered as bounded analytic functions (of the variable p) on the annulus $\{0 \leq v_\flat(\cdot) < \mu\}$.

It is clear that $A_\mu \subset A_{\mu'}$ (resp. $B_\mu^+ \subset B_{\mu'}^+$) as soon as $\mu \geq \mu'$. However, the reader should be careful that the functions $\mu \mapsto A_\mu$ and $\mu \mapsto B_\mu^+$ are not continuous in the sense that A_μ is strictly included in $\bigcap_{\mu'<\mu} A_{\mu'}$, and similarly for the B_μ^+'s. In the analytic language, a function in $\bigcap_{\mu'<\mu} B_{\mu'}^+$ is analytic on the annulus $\{0 \leq v_\flat(\cdot) < \mu\}$ but not necessarily bounded. Similarly $\bigcap_{\mu>0} B_\mu^+$ is strictly greater than the ring B_{inf}^+ we have introduced in §3.1; actually, we shall construct soon a quite important element t lying in the former ring but not in the latter. The relation between B_{crys}^+ and the B_μ^+'s is also simple to understand. Indeed, the estimation (27) shows that $B_\mu^+ \subset B_{\text{crys}}^+ \subset B_{p-1}^+$ for all $\mu > p-1$ (*cf* also Figure 4). At the integral level, we have $A_p \subset A_{\text{crys}} \subset A_{p-1}$.

For $\mu > 0$, we also define $A_{\mu,K} = \mathcal{O}_K \otimes_{W(k)} A_\mu$ and $B_{\mu,K}^+ = K \otimes_{K_0} B_\mu^+$. Elements in $B_{\mu,K}^+$ are series of the form $\sum_{i\in\mathbb{Z}} [\xi_i]\, \pi^i$ with $\lim_{i\to-\infty} v_\flat(\xi_i) + \frac{\mu i}{e} = +\infty$. The subring $A_{\mu,K}$ is characterized by the positivity condition $v_\flat(\xi_i) + \mu\lfloor \frac{i}{e} \rfloor \geq 0$ for all $i \in \mathbb{Z}$.

The notion of Newton polygons, which was defined for elements of B_{inf}^+ (resp. $B_{\text{inf},K}^+$) in §3.1, admits a straightforward extension to B_μ^+ (resp. $B_{\mu,K}^+$). Precisely, if $x = \sum_{i\in\mathbb{Z}} [\xi_i]\, p^i \in B_\mu^+$ (resp. $x = \sum_{i\in\mathbb{Z}} [\xi_i]\, \pi^i \in B_{\mu,K}^+$), we define $\text{NP}_\mu(x)$ as the convex hull of the points $(i, v_\flat(i))$ (resp. $(\frac{i}{e}, v_\flat(i))$) together with the two points at infinity $(0, +\infty)$ and $+\infty\cdot(-1, \mu)$. When $x \in \bigcap_{\mu>0} B_\mu^+$ (resp. $x \in \bigcap_{\mu>0} B_{\mu,K}^+$), we define $\text{NP}_{\text{inf}}(x) = \bigcap_{\mu>0} \text{NP}_\mu(x)$. One checks easily that this definition agrees with the definition of NP_{inf} on B_{inf}^+ (resp. on $B_{\text{inf},K}^+$) we gave earlier.

Finally, we observe that the Galois action and the Frobenius are well-defined on the A_μ's and B_μ^+'s. Even better, for all $\mu > 0$, the Frobenius induces isomorphisms of rings $A_\mu \to A_{p\mu}$, $B_\mu^+ \to B_{p\mu}^+$ and $B_{\mu,K}^+ \to B_{p\mu,K}^+$. As for Newton polygons, they are preserved under the action of G_K and we have the following transformation formula under Frobenius:

$$\text{NP}_{\mu p}(\varphi(x)) = \varphi_{\mathbb{R}^2}(\text{NP}_\mu(x)) \quad \text{where} \quad \varphi_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^2,\ (i, v) \mapsto (i, pv)$$

for $x \in B_{\mu,K}^+$. Passing to the limit on μ, we find that $\text{NP}_{\text{inf}}(\varphi(x)) = \varphi_{\mathbb{R}^2}(\text{NP}_{\text{inf}}(x))$ for all $x \in \bigcap_{\mu>0} B_{\mu,K}^+$.

[8] Otherwise, an element ξ with the required properties does not exist.

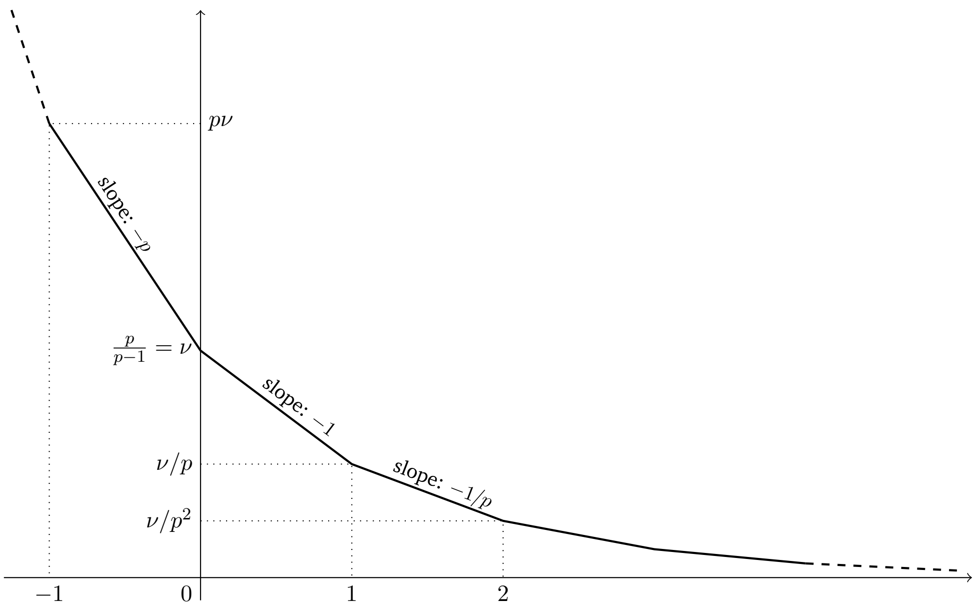

Figure 5: The Newton polygon of t

3.2.3 The element t

An essential property of A_{crys} is that it contains a period for the cyclotomic character, that is a special element on which Galois acts by multiplication by χ_{cycl}. This distinguished element is:

$$t = \log\,[\underline{\varepsilon}] = \sum_{i=1}^{\infty} (-1)^{i-1} \cdot \frac{([\underline{\varepsilon}]-1)^i}{i}.$$

Observe that the latter sum converges in A_{crys} since its i-th summand is equal to:

$$(-1)^{i-1} \cdot (i-1)! \cdot \big([\underline{\varepsilon}^{1/p}] - 1\big)^i \cdot \frac{\omega^i}{i!}$$

and therefore goes to 0 in A_{crys}, thanks to the factor $(i-1)!$ which converges to 0 for the p-adic topology. A similar computation shows that t actually lies in B^+_μ for all $\mu > 0$ and in A_μ for $\mu \geq 1 - \frac{1}{p}$.

Recall that the Frobenius and the group G_K act on $[\underline{\varepsilon}]$ by $\varphi([\underline{\varepsilon}]) = [\underline{\varepsilon}]^p$ and $g[\underline{\varepsilon}] = [\underline{\varepsilon}]^{\chi_{\text{cycl}}(g)}$ for $g \in G_K$. Taking logarithms, we find $\varphi(t) = pt$ and $gt = \chi_{\text{cycl}}(g)\, t$ for all $g \in G_K$. The latter relation is what we expected: the element t is a period for the cyclotomic character.

The Newton polygon of t can also be computed[9]. The result we find is displayed on Figure 5; we notice in particular that its slopes are unbounded, reflecting the fact that t is in B^+_μ for all $\mu > 0$. It also remains unchanged under the transformation $(i, v) \mapsto (i-1, pv)$, reflecting the fact that $\varphi(t) = pt$. In fact, the special shape of $\text{NP}_{\text{inf}}(t)$ is explained by the existence of a decomposition of t as an infinite convergent product (in all B_μ's), precisely:

$$t = \prod_{n=0}^{\infty} \varphi^{-n}(\omega) \cdot \prod_{n=1}^{\infty} \frac{\varphi^n(\omega)}{p} \tag{28}$$

[9]The computation can be carried out as follows. By Remark 3.1.6, we know that $\text{NP}_{\text{inf}}([\underline{\varepsilon}] - 1))$ is the set $\mathcal{P}^+$ defined as the convex hull of the points $A_n = \big(n, \frac{1}{(p-1)p^n}\big)$ for n varying in $\mathbb{N}$. By the multiplicativity property of Newton polygons, we find that $\text{NP}\big(\frac{([\underline{\varepsilon}]-1)^i}{i}\big) = \tau_{v_p(i)}(i\mathcal{P}^+)$ where τ_u is the translation of vector $(0, -u)$. We now observe that each A_n ($n \in \mathbb{Z}$) belongs to exactly one $\tau_{v_p(i)}(i\mathcal{P}^+)$: when $n \geq 0$, we have $i = 0$ and when $n < 0$, we have $i = p^{-n}$. Therefore the Newton polygon of t is the convex hull of the A_n's for n varying in $\mathbb{Z}$.

where $\omega = \frac{[\varepsilon]-1}{[\varepsilon]^{1/p}-1}$ is the element of Proposition 3.1.5. The factorization (28) should be paralleled with (24).

Definition 3.2.3. For $\mu > 0$ or $\mu = \text{crys}$, we set $B_\mu = B_\mu^+[\frac{1}{t}]$.

The Frobenius and the Galois action extend to B_μ without difficulty: for $g \in G_K$, $x \in B_\mu^+$ and $m \in \mathbb{N}$, we put $\varphi(\frac{x}{t^m}) = \frac{\varphi(x)}{p^m t^m}$ and $g(\frac{x}{t^m}) = \frac{gx}{\chi_{\text{cycl}}(g)^m \cdot t^m}$.

3.3 The de Rham filtration and the field B_{dR}

We recall that, in §3.1.3, we have constructed a ring homomorphism $\theta : B_{\text{inf}}^+ \to \mathbb{C}_p$, which was given by the explicit formula:

$$\theta : \quad \sum_{i=-\infty}^{\infty} [\xi_i]\, p^i \;\mapsto\; \sum_{i=-\infty}^{\infty} \xi_i^\sharp\, p^i \qquad (\xi_i \in \mathcal{R}),\ \xi_i = 0 \text{ for } i \ll 0).$$

The filtration by the power of the ideal $\ker\theta$ will play an important role because it will eventually correspond to the de Rham filtration on the cohomology. We devote this subsection to the study of its main properties. This will lead us to the definition of the period ring B_{dR}.

3.3.1 Definition and main properties of the de Rham filtration

First of all, we will need to extend the morphism θ to the rings B_μ's we have introduced earlier. Actually, just noticing that $v_p(\xi_i^\sharp p^i) = v_\flat(\xi_i) + i$, we deduce that θ extends readily to a ring homomorphism $\theta_\mu : B_\mu^+ \to \mathbb{C}_p$ whenever $\mu \geq 1$. Extending scalars to K, we obtain a ring homomorphism $\theta_{\mu,K} : B_{\mu,K}^+ \to \mathbb{C}_p$ for all $\mu \geq 1$. We observe that θ_μ (resp. $\theta_{\mu,K}$) maps the subring A_μ (resp. $A_{\mu,K}$) to $\mathcal{O}_{\mathbb{C}_p}$.

The condition $\mu \geq 1$ for the existence of θ_μ suggests that the rings A_1, $A_{1,K}$, B_1^+ and $B_{1,K}^+$ will play a particular role. In the literature, they are often denoted by A_{max}, $A_{\text{max},K}$, B_{max}^+ and $B_{\text{max},K}^+$ respectively; we will also use this notation in the sequel and will set $\theta_{\text{max}} = \theta_1$, $\theta_{\text{max},K} = \theta_{1,K}$ accordingly. Similarly, we will use the notation θ_{inf} and $\theta_{\text{inf},K}$ for θ and θ_K respectively.

We recall that A_{max} is the p-adic completion of $A_{\text{inf}}[\frac{[p^\flat]}{p}]$. In particular, we have canonical isomorphisms:

$$A_{\text{max}}/pA_{\text{max}} \simeq (\mathcal{R}/p^\flat\mathcal{R})[X] \simeq (\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p})[X], \quad [p^\flat]/p \leftarrow\!\shortmid X \tag{29}$$

the second isomorphism coming from Lemma 3.1.1. Similarly, we have:

$$A_{\text{max},K}/\pi A_{\text{max},K} \simeq (\mathcal{R}/\pi^\flat\mathcal{R})[X] \simeq (\mathcal{O}_{\mathbb{C}_p}/\pi\mathcal{O}_{\mathbb{C}_p})[X], \quad [\pi^\flat]/\pi \leftarrow\!\shortmid X. \tag{30}$$

We can also identify the kernels of θ_{max} and $\theta_{\text{max},K}$ (as we did for θ and θ_K in Proposition 3.1.3).

Proposition 3.3.1. *The ideal $\ker\theta_{\text{max}}$ (resp. $\ker\theta_{\text{max},K}$) is the principal ideal generated by the element $[p^\flat] - p$ (resp. $[\pi^\flat] - \pi$).*

Proof. We only give the proof for θ_{max}, the case of $\theta_{\text{max},K}$ being absolutely similar. We will prove that $1 - [p^\flat]/p$ is a generator of the ideal $A_{\text{max}} \cap \ker\theta_{\text{max}}$ of A_{max}. Let $\bar\theta_{\text{max}} : A_{\text{max}}/pA_{\text{max}} \to \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$ be the morphism induced by θ_{max}. Repeating the argument of Proposition 3.1.3, it is enough to show that $\ker\bar\theta_{\text{max}}$ is the principal ideal generated by $1 - [p^\flat]/p$. Under the isomorphism (29), $\bar\theta_{\text{max}}$ acts by the identity on $\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$ and takes X to 1. Hence, its kernel is the principal ideal generated by $X-1$. □

Definition 3.3.2. For $\mu \geq 1$, $\mu = \text{crys}$ or $\mu = \text{inf}$ and for $m \in \mathbb{N}$, we define:

$$\text{Fil}^m B_\mu^+ = (\ker\theta_\mu)^m \quad \text{and} \quad \text{Fil}^m B_{\mu,K}^+ = (\ker\theta_{\mu,K})^m.$$

We will use the notation gr to refer to the graded ring of a filtered ring: if m is an integer and $\mathfrak{A}$ is a filtered ring, we put $\mathrm{gr}^m\mathfrak{A} = \mathrm{Fil}^m\mathfrak{A}/\mathrm{Fil}^{m+1}\mathfrak{A}$ and $\mathrm{gr}\,\mathfrak{A} = \bigoplus_{m\geq 0}\mathrm{gr}^m\mathfrak{A}$. We recall that $\mathrm{gr}^0\mathfrak{A}$ is a ring and that $\mathrm{gr}^m\mathfrak{A}$ is a module over $\mathrm{gr}^0\mathfrak{A}$ for all $m \geq 0$. As for $\mathrm{gr}\,\mathfrak{A}$, is it a graded algebra over $\mathrm{gr}^0\mathfrak{A}$. In our case, we have $\mathrm{gr}^0 B^+_\mu = \mathrm{gr}^0 B^+_{\mu,K} = \mathbb{C}_p$ (since θ_μ and $\theta_{\mu,K}$ are surjective). Hence $\mathrm{gr}^m B^+_\mu$ and $\mathrm{gr}^m B^+_{\mu,K}$ have a natural structure of $\mathbb{C}_p$-vector space. They moreover inherit a Galois action, so that they are actually $\mathbb{C}_p$-semi-linear representations of G_K, *i.e.* objects of the category $\mathrm{Rep}_{\mathbb{C}_p}(G_K)$. From Proposition 3.1.3 and Proposition 3.3.1, we deduce that $\mathrm{gr}^m B^+_{\mathrm{inf}}$, $\mathrm{gr}^m B^+_{\mathrm{inf},K}$, $\mathrm{gr}^m B^+_{\max}$ and $\mathrm{gr}^m B^+_{\max,K}$ are all one dimensional over $\mathbb{C}_p$. As we shall see below (*cf* Proposition 3.3.4), this property also holds for $\mathrm{gr}^m B^+_\mu$ and $\mathrm{gr}^m B^+_{\mu,K}$ for any μ.

The next proposition shows that the de Rham filtration is separated.

Proposition 3.3.3. *For $\mu \geq 1$, $\mu = \mathrm{crys}$ or $\mu = \mathrm{inf}$, we have $\bigcap_m \mathrm{Fil}^m B^+_\mu = \bigcap_m \mathrm{Fil}^m B^+_{\mu,K} = 0$.*

Proof. Since B^+_μ and $B^+_{\mu,K}$ contain $B^+_{\max,K}$, it is enough to prove the proposition for $\theta_{\max,K}$. After Proposition 3.3.1, we are reduced to check that if $x \in B^+_{\max,K}$ is divisible by $\big(1 - \frac{[\pi^\flat]}{\pi}\big)^m$ for all m, then $x = 0$. Multiplying x by the adequate power of π, we may assume that $x \in A_{\max,K}$ and in addition, if $x \neq 0$, that $x \notin \pi A_{\max}$. Using isomorphism (29), we find that x vanishes in $A_{\max,K}/\pi A_{\max,K}$, *i.e.* $x \in \pi A_{\max,K}$. By our assumption, this implies that $x = 0$. □

Proposition 3.3.4. *For $\mu \geq 1$ or $\mu = \mathrm{crys}$ and for $m \in \mathbb{N}$, the inclusion $B^+_{\mathrm{inf}} \to B^+_\mu$ (resp. $B^+_{\mathrm{inf},K} \to B^+_{\mu,K}$) induces a G_K-equivariant isomorphism*

$$B^+_{\mathrm{inf}}/\mathrm{Fil}^m B^+_{\mathrm{inf}} \simeq B^+_\mu/\mathrm{Fil}^m B^+_\mu \qquad (\text{resp. } B^+_{\mathrm{inf},K}/\mathrm{Fil}^m B^+_{\mathrm{inf},K} \simeq B^+_{\mu,K}/\mathrm{Fil}^m B^+_{\mu,K}).$$

Proof. In the case $\mu = 1$, the proposition follows by combining Propositions 3.1.3 and 3.3.1. Before moving to the case of a general μ, we will prove an additional continuity property of the isomorphism $B^+_{\mathrm{inf}}/\mathrm{Fil}^m B^+_{\mathrm{inf}} \simeq B^+_{\max}/\mathrm{Fil}^m B^+_{\max}$, which will be useful later. Precisely, we claim that, for all $m \geq 0$, there exists a nonnegative integer v_m such that:

$$p^{v_m} \cdot A_{\max}/\mathrm{Fil}^m A_{\max} \subset A_{\mathrm{inf}}/\mathrm{Fil}^m A_{\mathrm{inf}} \tag{31}$$

We prove the claim by induction on m. For $m = 0$, there is nothing to prove (we can take $v_0 = 0$). We now assume that (31) is proved for m. We consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{gr}^m A_{\mathrm{inf}} & \longrightarrow & A_{\mathrm{inf}}/\mathrm{Fil}^{m+1} A_{\mathrm{inf}} & \longrightarrow & A_{\mathrm{inf}}/\mathrm{Fil}^m A_{\mathrm{inf}} & \longrightarrow & 0 \\
 & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\
0 & \longrightarrow & \mathrm{gr}^m A_{\max} & \longrightarrow & A_{\max}/\mathrm{Fil}^{m+1} A_{\max} & \longrightarrow & A_{\max}/\mathrm{Fil}^m A_{\max} & \longrightarrow & 0.
\end{array}$$

From the fact that $\mathrm{gr}^m B^+_{\mathrm{inf}} \to \mathrm{gr}^m B^+_{\max}$ is a $\mathbb{C}_p$-linear mapping between two one-dimensional $\mathbb{C}_p$-vector spaces, we deduce that there exists an integer v such that $p^v{\cdot}\mathrm{gr}^m A_{\max} \subset \mathrm{gr}^m A_{\mathrm{inf}}$. A diagram chase then shows that (31) holds with $v_{m+1} = v_m + v$.

We now go back to the proof of the proposition. We pick $\mu \in (1,+\infty) \sqcup \{\mathrm{crys}\}$ and $m \in \mathbb{N}$. Let $f : B^+_{\mathrm{inf}}/(\ker\theta)^m \to B^+_\mu/(\ker\theta_\mu)^m$ be the morphism of the proposition. Let A'_μ be the sub-A_{inf}-algebra of B^+_{inf} generated by all the elements of the form $\frac{[\xi]}{p^i}$ ($\xi \in \mathcal{R}$, $i \in \mathbb{N}$), which belong to A_μ. Then $A'_\mu \subset B^+_{\mathrm{inf}}$ and A_μ appears as the p-adic completion of A'_μ. The former property implies that we have a morphism $g' : A'_\mu/(\ker\theta_\mu)^m \to B^+_{\mathrm{inf}}/(\ker\theta)^m$. We claim that g' is continuous. Indeed, by (31), we have $A_{\max} \subset p^{-v_m}A_{\mathrm{inf}} + (\ker\theta_{\max})^m$. Since $A'_\mu \subset A_{\max}$, we deduce that g' maps $A'_\mu/(\ker\theta_\mu)^m$ to $p^{-v_m}A_{\mathrm{inf}}/(\ker\theta)^m$, which implies its continuity. Now, passing to the p-adic completion and inverting p, we find that g' induces a ring morphism $g : B^+_\mu/(\ker\theta_\mu)^m \to B^+_{\mathrm{inf}}/(\ker\theta)^m$, which is an inverse of f. Therefore f is an isomorphism.

The identification $B^+_{\mathrm{inf},K}/\mathrm{Fil}^m B^+_{\mathrm{inf},K} \simeq B^+_{\mu,K}/\mathrm{Fil}^m B^+_{\mu,K}$ is obtained similarly. □

Proposition 3.3.4 implies that for all m, all the maps of the commutative square below are isomorphisms of $\mathbb{C}_p$-semi-linear representations:

$$\begin{array}{ccc} \mathrm{gr}^m B^+_{\mathrm{inf}} & \xrightarrow{\sim} & \mathrm{gr}^m B^+_{\mu} \\ \sim\downarrow & & \downarrow\sim \\ \mathrm{gr}^m B^+_{\mathrm{inf},K} & \xrightarrow{\sim} & \mathrm{gr}^m B^+_{\mu,K} \end{array} \tag{32}$$

We can moreover entirely elucidate the Galois action. Indeed we have the following proposition.

Proposition 3.3.5. *For $\mu \geq 1$ or $\mu = \mathrm{crys}$ and for $m \in \mathbb{N}$, the spaces $\mathrm{gr}^m B^+_{\mu}$ and $\mathrm{gr}^m B^+_{\mu,K}$ are generated by the class of t^m.*

Proof. Thanks to the diagram (32), it is enough to prove the proposition for $\mathrm{gr}^m B^+_{\mu}$. We already know that $\mathrm{gr}^m B^+_{\mu}$ is one dimensional over $\mathbb{C}_p$. We observe that t lies in $\ker \theta_\mu$ since $\theta_\mu(t) = \log \theta_\mu([\underline{\varepsilon}]) = \log 1 = 0$. Thus $t^m \in \mathrm{Fil}^m B^+_{\mu}$ and we are reduced to prove that t^m is not zero in $\mathrm{gr}^{m+1} B^+_{\mu}$. Noting that $t \equiv [\underline{\varepsilon}] - 1 \pmod{\mathrm{Fil}^2 B^+_{\mu}}$, we can replace t by $[\underline{\varepsilon}] - 1$. Using again the diagram (32), it is enough to show that $([\underline{\varepsilon}]-1])^m$ does not vanish in $\mathrm{gr}^{m+1} B^+_{\mathrm{inf}}$. Let ω be the element of Proposition 3.1.5, so that we can write $([\underline{\varepsilon}] - 1)^m = \omega^m \cdot ([\underline{\varepsilon}]^{1/p} - 1)^m$. We know that the class of ω^m is a generator of $\mathrm{gr}^m B^+_{\mathrm{inf}}$. It is enough to check θ does not vanish on $([\underline{\varepsilon}]^{1/p} - 1)^m$. But a direct computation gives $\theta(([\underline{\varepsilon}]^{1/p} - 1)^m) = (\varepsilon_1 - 1)^m$ where $\varepsilon_1 \in \mathbb{C}_p$ is a primitive p-th root of unity. We conclude by noticing that $\varepsilon_1 \neq 1$. □

Remark 3.3.6. We strongly insist on the fact that t^m is *not* a generator of $\mathrm{Fil}^m B^+_{\mu}$ (resp. $\mathrm{Fil}^m B^+_{\mu,K}$) since this is often the source of confusion. Let us clarify this point by examining a bit the case where $\mu = 1$. Then, by Proposition 3.3.1, we know that $\mathrm{Fil}^1 B^+_{\mathrm{max}}$ is generated by the element $\gamma = [p^\flat] - p$. Thus, we can write $t = \gamma\gamma'$ for some $\gamma' \in B^+_{\mathrm{max}}$. It turns out that γ' is not invertible in B^+_{max} but is a unit in $B^+_{\mathrm{max}}/\mathrm{Fil}^1 B^+_{\mathrm{max}}$ (which is isomorphic to $\mathbb{C}_p$), reflecting the fact that t^m does not generate $\mathrm{Fil}^m B^+_{\mathrm{max}}$ but generates $\mathrm{gr}^m B^+_{\mathrm{max}}$. The situation is quite similar to the following one which is very familiar to the number theorists: pick an odd prime number p, equip $\mathbb{Z}$ with the filtration $\mathrm{Fil}^m \mathbb{Z} = p^m \mathbb{Z}$ and consider the element $t = 2p$. Then t^m is not a generator of $\mathrm{Fil}^m \mathbb{Z}$ but it does generate $\mathrm{gr}^m \mathbb{Z}$ because 2 is invertible modulo p.

It follows from Proposition 3.3.5 that $\mathrm{gr}^m B^+_{\mu}$ and $\mathrm{gr}^m B^+_{\mu,K}$ are both isomorphic to $\mathbb{C}_p(\chi^m_{\mathrm{cycl}})$ in the category $\mathrm{Rep}_{\mathbb{C}_p}(G_K)$. Passing to the graduation, we obtain G_K-equivariant isomorphisms of rings:

$$\mathrm{gr}\, B^+_{\mu} \simeq \mathrm{gr}\, B^+_{\mu,K} \simeq \mathbb{C}_p[t] \qquad (\mu \in [1, +\infty) \sqcup \{\mathrm{inf}, \mathrm{crys}, \mathrm{max}\}) \tag{33}$$

where the letter t on the right hand side is a new variable (corresponding to the special element $t \in B^+_{\mu}$) on which Galois acts by multiplication by the cyclotomic character.

3.3.2 Completion with respect to the de Rham filtration

After what we have done previously, it is natural to introduce the completion of the B^+_{μ}'s (resp. the $B^+_{\mu,K}$'s) with respect to the de Rham filtration. This actually leads to the definition of the period ring B^+_{dR}.

Definition 3.3.7. We define B^+_{dR} as the completion of B^+_{inf} for the $(\ker \theta)$-adic topology:

$$B^+_{\mathrm{dR}} = \varprojlim_m B^+_{\mathrm{inf}}/(\ker \theta)^m = \varprojlim_m B^+_{\mathrm{inf}}/\mathrm{Fil}^m B^+_{\mathrm{inf}}.$$

Since each quotient $B^+_{\mathrm{inf}}/\mathrm{Fil}^m B^+_{\mathrm{inf}}$ has a Galois action, B^+_{dR} inherits an action of G_K. Besides, the algebraic structure of B^+_{dR} is very pleasant. Indeed, from the fact that $\ker \theta$ is a principal ideal of B^+_{inf}, we deduce that B^+_{dR} is a discrete valuation ring. Its maximal ideal is the ideal generated by $\ker\theta$ and its residue field is canonically isomorphic to $B^+_{\text{inf}}/\text{Fil}^1 B^+_{\text{inf}} \simeq \mathbb{C}_p$. Therefore, as a ring, B^+_{dR} is isomorphic $\mathbb{C}_p(\!(t)\!)$, that is to the ring B'_{HT} we introduced in §2.2.3. However, we strongly insist on the fact that there is no such isomorphism preserving the Galois action. The sole connection between B^+_{dR} and $\mathbb{C}_p(\!(t)\!)$ is that they share the same graded ring, namely B_{HT}.

Observe that, by Proposition 3.3.4, we could have defined alternatively B^+_{dR} as the completion of B^+_μ or $B^+_{\mu,K}$, *i.e.* we have the following canonical identifications:

$$B^+_{\text{dR}} = \varprojlim_m B^+_\mu/\text{Fil}^m B^+_\mu = \varprojlim_m B^+_{\mu,K}/\text{Fil}^m B^+_{\mu,K}.$$

for any $\mu \in [1,+\infty) \sqcup \{\text{inf}, \text{crys}, \text{max}\}$. Combining this with the fact that the de Rham filtration is separated (*cf* Proposition 3.3.3), we deduce that the canonical maps $B^+_\mu \to B^+_{\mu,K} \to B^+_{\text{dR}}$ are injective for all μ as before. In particular $t \in B^+_{\text{dR}}$ and B^+_{dR} contains a copy of K. Since the definition of B^+_{dR} does not actually depend on K, it follows that B^+_{dR} contains (in a coherent way) a copy of any finite extension of $\mathbb{Q}_p$, that is a copy of $\bar{K}$. Denote by $\iota : \bar{K} \to B^+_{\text{dR}}$ the resulting embedding. It turns out that ι can be understood in more down-to-earth terms. Indeed observe first that $\bar{K}$ naturally embeds into the residue field of B^+_{dR} since the latter is canonically isomorphic to $\mathbb{C}_p$. By Hensel lemma, this embedding admits a unique lifting $\iota : \bar{K} \to B^+_{\text{dR}}$ which is a homomorphism of K_0-algebras: concretely, for $x \in \bar{K}$ whose minimal polynomial over K_0 is denoted by P, $\iota(x)$ is the unique root of P that lifts the image of x in $\mathbb{C}_p$. In particular, the composite $\bar{K} \to B^+_{\text{dR}} \to \mathbb{C}_p$ is the natural inclusion.

The map θ extends to B^+_{dR} easily: we define θ_{dR} as the composite $B^+_{\text{dR}} \to B^+_{\text{inf}}/(\ker\theta) \to \mathbb{C}_p$ where the first map is the projection onto the first component and the second map is induced by θ. We set $\text{Fil}^m B^+_{\text{dR}} = (\ker\theta_{\text{dR}})^m$ for $m \in \mathbb{N}$. Observe that the kernel of θ_{dR} is nothing but the maximal ideal of B^+_{dR}. As a consequence, the de Rham filtration of B^+_{dR} coincides with the canonical filtration on the discrete valuation ring B^+_{dR}, given by the valuation. Its graded ring is isomorphic to $\mathbb{C}_p[t]$ (compare with (33)). Moreover, any generator of B^+_μ or $B^+_{\mu,K}$ (for $\mu \in \{\text{inf}, \text{max}\}$) is a generator of B^+_{dR}, *i.e.* a uniformizer of B^+_{dR}. Even better, by completeness, an element of B^+_{dR} is a uniformizer if and only if it does not belong to $\text{Fil}^1 B^+_{\text{dR}}$ or, equivalently, it does not vanish if $\text{gr}^1 B^+_{\text{dR}}$. In particular, the special element t is a uniformizer of B^+_{dR} by Proposition 3.3.5.

Remark 3.3.8. Continuing Remark 3.3.6 (and importing notations from there), we observe that the element $\gamma' \in B^+_{\text{max}} \subset B^+_{\text{dR}}$ is invertible in B^+_{dR} since it is nonzero in the residue field; thus $t = \gamma\gamma'$ is a generator of $\ker\theta_{\text{dR}}$ as γ is. This contrasts with the fact that t did not generate $\ker\theta_{\text{max}}$ because γ' was not invertible in B^+_{max}.

Topology on B^+_{dR}. As B^+_{dR} is defined as a completion, the first natural topology on B^+_{dR} is the $(\ker\theta)$-adic topology: a sequence $(x_n)_{n\geq 0}$ of elements of B^+_{dR} converges to $x \in B^+_{\text{dR}}$ if and only if, for all m, the sequence $x_n \bmod \text{Fil}^m B^+_{\text{dR}}$ is eventually constant. This topology is actually not nice because it does not see the p-adic topology: it induces the discrete topology both on the subfield $\bar{K}\cdot\hat{K}^{\text{ur}}$ and on the residue field $\mathbb{C}_p$.

A coarser topology can be defined as follows. Observe that the quotients $B^+_{\text{inf}}/\text{Fil}^m B^+_{\text{inf}}$ have finite length and hence are equipped with a canonical topology. This topology can be described by remarking that the lattice $A_{\text{inf}}/\text{Fil}^m A_{\text{inf}}$ defines a valuation $v_{m,\text{inf}}$ on $B^+_{\text{dR}}/\text{Fil}^m B^+_{\text{dR}}$: given $x \in B^+_{\text{dR}}/\text{Fil}^m B^+_{\text{dR}} \simeq B^+_{\text{inf}}/\text{Fil}^m B^+_{\text{inf}}$, we define $v_{m,\text{inf}}(x)$ as the largest (possible negative) integer n for which $x \in p^n A_{\text{inf}}/\text{Fil}^m A_{\text{inf}}$. The valuation $v_{m,\text{inf}}$ defines a norm on $B^+_{\text{inf}}/\text{Fil}^m B^+_{\text{inf}}$, and hence a topology.

Remark 3.3.9. Alternatively, instead of B^+_{inf}, one could have worked with B^+_μ for a different μ. We would have ended up this way with a valuation $v_{m,\mu}$ on $B^+_{\text{dR}}/\text{Fil}^m B^+_{\text{dR}}$ for which there exists a constant $v_{m,\mu}$ with the property that:

$$v_{m,\inf}(x) - v_{m,\mu} \leq v_{m,\mu}(x) \leq v_{m,\inf}(x) \tag{34}$$

for all $x \in B^+_{\mathrm{dR}}/\mathrm{Fil}^m B^+_{\mathrm{dR}}$ (see the first part of the proof of Proposition 3.3.4). Therefore the topology induced by $v_{m,\mu}$ agrees with that defined by $v_{m,\inf}$ for all m.

We extend $v_{m,\inf}$ to B^+_{dR} by precomposing by the natural projection $B^+_{\mathrm{dR}} \to B^+_{\mathrm{dR}}/\mathrm{Fil}^m B^+_{\mathrm{dR}}$. When m varies, the $v_{m,\inf}$'s define a family of semi-norms on B^+_{dR}, giving it the structure of a Frechet space. The attached topology will be called (in this article) the *standard topology* on B^+_{dR}. Concretely, a sequence (x_n) of elements of B^+_{dR} converges to $x \in B^+_{\mathrm{dR}}$ for the standard topology if and only if, for all integer m, the image of x_n is $B^+_{\mathrm{dR}}/\mathrm{Fil}^m B^+_{\mathrm{dR}} \simeq B^+_{\mathrm{inf}}/\mathrm{Fil}^m B^+_{\mathrm{inf}}$ converges to the image of x. Clearly, the standard topology induces the usual p-adic topology on the residue field $B^+_{\mathrm{dR}}/\mathrm{Fil}^1 B^+_{\mathrm{dR}} \simeq \mathbb{C}_p$. Colmez proved in [10] that $\bar{K}$ is dense in B^+_{dR} for the standard topology.

We point out that there is no good notion of p-adic topology on B^+_{dR}. Indeed, if there were, the inclusion $\iota : \bar{K} \to B^+_{\mathrm{dR}}$ would extend to an inclusion $\mathbb{C}_p \to B^+_{\mathrm{dR}}$ which would imply that B^+_{dR} would be isomorphic to $\mathbb{C}_p((t)) = B'_{\mathrm{HT}}$ and we have already seen that this does not happen. Yet, B^+_{dR} admits kinds of lattices, *e.g.*

$$A_{\mu,\mathrm{dR}} = \varprojlim_m A_\mu/(A_\mu \cap \mathrm{Fil}^m B^+_\mu) \quad \text{or} \quad A_{\mu,K,\mathrm{dR}} = \varprojlim_m A_{\mu,K}/(A_{\mu,K} \cap \mathrm{Fil}^m B^+_{\mu,K})$$

though we have to be careful that $A_{\mu,\mathrm{dR}}[\frac{1}{p}] \subsetneq B_{\mathrm{dR}}$ and similarly for $A_{\mu,K,\mathrm{dR}}$. These "lattices" do define topologies on B^+_{dR} (which might be considered as sort of p-adic topologies). However, these topologies are all different (and different from the standard topology) and they all have bad properties; for instance, the inclusion $\iota : \bar{K} \to B^+_{\mathrm{dR}}$ is not continuous for any of them. The point behind this is that the constant $v_{m,\mu}$ of Eq. (34) is not bounded uniformly when m grows.

Remark 3.3.10. The situation is quite similar to that $\mathbb{Q}_p[[t]]$. The analogue of the standard topology on $\mathbb{Q}_p[[t]]$ is the standard Fréchet topology on this ring: a sequence $(f_n)_{n\geq 0}$ converges to f if and only if $f_n \bmod t^m$ converges to $f \bmod t^m$ in $\mathbb{Q}_p[t]/t^m$ for all $m \in \mathbb{N}$. This is further equivalent to the fact that, for all fixed $m \in \mathbb{N}$, the m-th coefficient of f_n converges to the m-th coefficient of f. Another topology on $\mathbb{Q}_p[[t]]$ is that defined by the "lattice" $\mathbb{Z}_p[[t]]$, for which the sequence $(f_n)_{n\geq 0}$ converges to f when, for each $A \geq 0$, there exists an index n_0 with the property that $f_n \equiv f \pmod{p^A \mathbb{Z}_p[[x]]}$ for all $n \geq n_0$. This notion of convergence is stronger than the previous one because we impose here that the coefficients of f_n converge *uniformly* to that of f (the index n_0 has to the same for all m).

Inverting t. Recall that we have defined B_μ and $B_{\mu,K}$ as $B^+_\mu[\frac{1}{t}]$ and $B^+_{\mu,K}[\frac{1}{t}]$ respectively for $\mu \geq 1$ or $\mu = \mathrm{crys}$ (recall that this definition does not make sense for $\mu = \inf$ because $t \notin B^+_{\mathrm{inf}}$). Similarly we set $B_{\mathrm{dR}} = B^+_{\mathrm{dR}}[\frac{1}{t}]$. Since B^+_{dR} is a discrete valuation ring with uniformizer t, B_{dR} is also the fraction field of B^+_{dR}; in particular, it is a field. Moreover since localization is exact, the rings B_μ and $B_{\mu,K}$ appear as subrings of B_{dR}.

The de Rham filtration extends readily to B_{dR} by letting $\mathrm{Fil}^m B_{\mathrm{dR}} = t^m B^+_{\mathrm{dR}}$ for $m \in \mathbb{Z}$. The graded ring of B_{dR} is then canonically isomorphic to $\mathbb{C}_p[t, t^{-1}] = B_{\mathrm{HT}}$. If B is any subring of B_{dR}, we define:

$$\mathrm{Fil}^m B = B \cap \mathrm{Fil}^m B_{\mathrm{dR}} \qquad (m \in \mathbb{Z}). \tag{35}$$

Observe that $\mathrm{Fil}^0 B$ is the intersection of two rings and thus is a ring as well. It is easily checked that, when $B = B^+_\mu$ or $B^+_{\mu,K}$ (for $\mu \geq 1$ or $\mu = \mathrm{crys}$), the above definition leads to the de Rham filtration $\mathrm{Fil}^m B$ we have defined earlier by different means. Yet, the definition (35) is new and interesting for $B = B_\mu$ and $B = B_{\mu,K}$. The filtrations obtained this way sit in the following diagram (and a similar diagram for $B_{\mu,K}$):

$$\begin{array}{ccccccccccccc} \cdots & \subset & \mathrm{Fil}^2 B_\mu & \subset & \mathrm{Fil}^1 B_\mu & \subset & \mathrm{Fil}^0 B_\mu & \subset & \mathrm{Fil}^{-1} B_\mu & \subset & \mathrm{Fil}^{-2} B_\mu & \subset & \cdots \\ & & \cup & & \cup & & \cup & & & & & & \\ \cdots & \subset & \mathrm{Fil}^2 B_\mu^+ & \subset & \mathrm{Fil}^1 B_\mu^+ & \subset & \mathrm{Fil}^0 B_\mu^+ = B_\mu^+ & & & & & & \end{array}$$

The reader should be very careful that the inclusion $B_\mu^+ \subset \mathrm{Fil}^0 B_\mu$ is strict. Let us first focus on the case where $\mu = \max$. We recall that, in Remark 3.3.6, we have set $\gamma = [p^\flat] - p \in B_{\max}^+$ and noticed that $t = \gamma\gamma'$ for some $\gamma' \in B_{\max}^+$. The element γ' is not invertible in $B_{\max}^+$ but we have seen in Remark 3.3.8 that it is invertible in B_{dR}^+. Besides, since γ' is a divisor of t, it is invertible in $B_{\max}$. Now consider $\frac{1}{\gamma'} \in B_{\max}$. It does not lie in $B_{\max}^+$. However, its image in B_{dR} falls in B_{dR}^+, so that $\frac{1}{\gamma'} \in \mathrm{Fil}^0 B_{\max}$. Actually, one can (easily) prove that $\mathrm{Fil}^0 B_{\max} = B_{\max}^+[\frac{1}{\gamma'}]$. A similar description is also possible for a general μ. Precisely let S be the multiplicative part consisting of all divisors in B_μ^+ of some power of t. Then $\mathrm{Fil}^0 B_\mu = B_\mu^+[S^{-1}]$.

Remark 3.3.11. As discussed in Remark 3.3.6, what happens here is very similar to the following very classical situation: assume that $\mathbb{Z}$ is endowed with the filtration $\mathrm{Fil}^m \mathbb{Z} = p^m \mathbb{Z}$, which induces the usual valuation filtration on $\mathbb{Z}_p$ and $\mathbb{Q}_p$ after completion. Now consider the localization $\mathbb{Z}[\frac{1}{2p}]$; it is a subring of $\mathbb{Q}_p$ and then inherits the valuation filtration. For this filtration, we have $\mathrm{Fil}^0\, \mathbb{Z}[\frac{1}{2p}] = \mathbb{Z}[\frac{1}{2}]$.

Remark 3.3.12. The reader may wonder why we defined B_μ as $B_\mu^+[\frac{1}{t}]$ and not $B_\mu^+[\frac{1}{\gamma}]$ in order to avoid the small unpleasantness discussed above. One reason is that the Frobenius does not extend on $B_\mu^+[\frac{1}{\gamma}]$ because the ideal $\ker \theta_\mu$ is not stable under Frobenius. Formula (28) shows that inverting t is very natural if our objective is to keep an action of the Frobenius.

Proposition 3.3.13. *For $\mu \geq 1$ or $\mu = \mathrm{crys}$, the inclusions $B_\mu \subset B_{\mu,K} \subset B_{\mathrm{dR}}$ induce G_K-equivariants isomorphisms of rings $\mathrm{gr}\, B_\mu \simeq \mathrm{gr}\, B_{\mu,K} \simeq \mathrm{gr}\, B_{\mathrm{dR}} \simeq B_{\mathrm{HT}}$.*

Proof. The fact that $\mathrm{gr}\, B_{\mathrm{dR}}$ is isomorphic to B_{HT} has been already noticed. Now, consider the composite $f : \mathrm{gr}\, B_\mu^+ \to \mathrm{gr}\, B_\mu \to \mathrm{gr}\, B_{\mu,K} \to \mathrm{gr}\, B_{\mathrm{dR}} \simeq \mathrm{gr}\, B_{\mathrm{dR}}^+$ in which all maps are injective. Since B_{dR}^+ is the completing of B_μ^+ with respect to the de Rham filtration, the map f has to be an isomorphism. The proposition follows. □

3.4 B_{crys} and B_{dR} as period rings

In order to apply Fontaine's general strategy (discussed in §1.4) with the B_μ's (for $\mu \geq 1$ or $\mu = \mathrm{crys}$ or $\mu = \mathrm{dR}$)—and then "promote" these rings at the level of genuine period rings—a final couple of verifications still need to be done; precisely we need to check that the B_μ's satisfy Fontaine's hypotheses (H1), (H2) and (H3) introduced in §1.4.1, and we need to compute the invariants under the G_K-action.

We start with B_{dR} which is easier. First, since it is a field, Fontaine's hypotheses are obviously fulfilled. Concerning the computation of the fixed points, we have the following theorem.

Theorem 3.4.1. *We have $(B_{\mathrm{dR}})^{G_K} = K$.*

Proof. We have already seen that K embeds into B_{dR}, so that $K \subset (B_{\mathrm{dR}})^{G_K}$. The reverse inclusion follows from the fact that $(\mathrm{gr}\, B_{\mathrm{dR}})^{G_K} = (B_{\mathrm{HT}})^{G_K} = K$. □

We now move to the crystalline setting, that is the ring B_{crys} and its variant B_μ with $\mu \geq 1$.

Theorem 3.4.2. *For $\mu \geq 1$ or $\mu = \mathrm{crys}$, we have $(B_\mu)^{G_K} = K_0$ and $(B_{\mu,K})^{G_K} = K$.*

Proof. We have already seen that $K_0 \subset (B_\mu)^{G_K}$ and $K \subset (B_{\mu,K})^{G_K}$. From $B_{\mu,K} \subset B_{\mathrm{dR}}$, we deduce that $(B_{\mu,K})^{G_K} \subset (B_{\mathrm{dR}})^{G_K} = K$, the latter equality resulting from Theorem 3.4.1. Hence we have proved that $(B_{\mu,K})^{G_K} = K$. Now remember that, by definition, $B_{\mu,K} = K \otimes_{K_0} B_\mu$. Taking the G_K-invariants, we obtain $K = K \otimes_{K_0} (B_\mu)^{G_K}$, from which we deduce $(B_\mu)^{G_K} = K_0$. □

Proposition 3.4.3. *For $\mu \geq 1$ or $\mu = \text{crys}$, the rings B_μ and $B_{\mu,K}$ satisfy Fontaine's hypotheses.*

Proof. It is clear that B_μ and $B_{\mu,K}$ are domains since they both embed into B_{dR} which is a field. Repeating the proof of Theorem 3.4.2, we find that $(\text{Frac}\, B_\mu)^{G_K} = K_0$ and $(\text{Frac}\, B_{\mu,K})^{G_K} = K$. Hence B_μ and $B_{\mu,K}$ satisfy hypothesis (H2).

Let us now prove that B_μ satisfies Fontaine's hypothesis (H3). Let $x \in B_\mu$, $x \neq 0$ and assume that the line $\mathbb{Q}_p x$ is stable under the action of G_K. We have to prove that x is invertible in B_μ. In what follows, we will consider x as an element on B_{dR}. Replacing possibly x by $t^n x$ for some integer n (which is safe since t is invertible in B_μ), we may assume that $x \in B_{\text{dR}}^+$ and $x \notin \text{Fil}^1 B_{\text{dR}}^+$. The morphism θ_{dR} then induces a G_K-equivariant embedding $\mathbb{Q}_p x \hookrightarrow \mathbb{C}_p$. Thus the representation $\mathbb{Q}_p x$ is $\mathbb{C}_p$-admissible. By Theorem 2.2.1, the inertia subgroup I_K of G_K acts on x through a finite quotient. Therefore there exists a positive integer n such that I_K acts trivially on $y = x^n$. The line $\mathbb{Q}_p y$ then inherits an action of $G_K/I_K = \text{Gal}(K^{\text{ur}}/K) \simeq \text{Gal}(K_0^{\text{ur}}/K_0)$. Applying Proposition 2.2.5 to the $\text{Gal}(K_0^{\text{ur}}/K_0)$-representation $\hat{K}_0^{\text{ur}} y$ (recall that $\hat{K}_0^{\text{ur}} \subset B_{\text{inf}}^+ \subset B_{\text{dR}}$), we find that there exists $\lambda \in \hat{K}_0^{\text{ur}}$ such that λy is fixed by G_K. By Theorem 3.4.2, we obtain $\lambda y \in K_0$ and then $y \in \hat{K}_0^{\text{ur}}$. We deduce that y is invertible in B_μ, and so also is x.

The fact that $B_{\mu,K}$ satisfies (H3) is proved in a similar fashion. □

We conclude this section by stating another important property of the rings B_μ.

Proposition 3.4.4. *Let $\mu \geq 1$ and $\mu = \text{crys}$. Let $x \in \text{Fil}^0 B_\mu$ such that $\varphi(x) = x$, then $x \in \mathbb{Q}_p$.*

Proof. We first prove the proposition when $\mu = \mu_0 = \frac{p}{p-1}$. In this case, it is easily checked that A_{μ_0} is the p-adic completion of $A_{\text{inf}}[\frac{t}{p}]$. This implies that $A_{\mu_0} \subset A_{\text{inf}} + \frac{t}{p} A_{\mu_0}$ and thus, inverting p, we find $B_{\mu_0}^+ \subset B_{\text{inf}}^+ + t B_{\mu_0}^+$.

Let $x \in \text{Fil}^0 B_{\mu_0}$ such that $\varphi(x) = x$. By definition of B_{μ_0}, we can write $x = t^{-m} y$ with $m \in \mathbb{N}$ and $y \in B_{\mu_0}^+$. We choose m minimal with this property. We assume by contradiction that $m > 0$. By the first paragraph of the proof, we can write $y = a + tb$ with $a \in B_{\text{inf}}^+$ and $b \in B_{\mu_0}^+$. Besides, for any nonnegative integer n, we have $\varphi^n(y) = p^{nm} y$ and then:

$$\theta \circ \varphi^n(a) = \theta \circ \varphi^n\big(y - tb\big) = \theta\big(p^{mn} y - p^n t \varphi^n(b)\big) = p^{mn} \cdot \theta_{\mu_0}(y) = 0$$

the last equality coming from the fact that $y = t^m x \in \text{Fil}^m B_{\mu_0} \subset \text{Fil}^1 B_{\mu_0}$. By Proposition 3.1.7, we find that $[\underline{\varepsilon}] - 1$ divides a in B_{inf}^+. On the other hand, from the definition of t, we have:

$$pt = \varphi(t) = \big([\underline{\varepsilon}]^p - 1\big) \cdot \sum_{i=1}^{\infty} (-1)^{i-1} \frac{([\underline{\varepsilon}]^p - 1)^{i-1}}{i},$$

from what we derive that t and $[\underline{\varepsilon}]^p - 1$ differ by a unit in $B_{\mu_0}^+$. From the divisibility observed above, we deduce that $[\underline{\varepsilon}]^p - 1$ divides $\varphi(a) = p^m a + p^m tb - pt\varphi(b)$ in $B_{\mu_0}^+$. Therefore t must divide a in $B_{\mu_0}^+$, which contradicts the minimality of m. As a conclusion, we find $m = 0$, *i.e.* $x \in B_{\mu_0}^+$.

Write $x = a + tb$ with $a \in B_{\text{inf}}^+$ and $b \in B_{\mu_0}^+$. The equality $x = \varphi(x)$ gives $x = \varphi^n(a) + p^n t \varphi^n(b)$ for all n. Therefore $\varphi^n(a)$ converges to x when n goes to infinity. Since B_{inf}^+ is closed in $B_{\mu_0}^+$, we deduce that $x \in B_{\text{inf}}^+$. Finally, remembering that $B_{\text{inf}}^+ = W(\mathcal{R})[\frac{1}{p}]$, we obtain $x \in W(\mathbb{F}_p)[\frac{1}{p}]$, that is $x \in \mathbb{Q}_p$.

We now move to the general case. Let $x \in (\text{Fil}^0 B_\mu)^{\varphi=1}$. In particular $x \in \text{Fil}^0 B_{\text{max}}$ and therefore $x = \varphi(x) \in \text{Fil}^0 B_p \subset \text{Fil}^0 B_{\mu_0}$. The conclusion now follows by the first part of the proof. □

Remark 3.4.5. Proposition 3.4.4 can be written in the shorter form:

$$\big(\text{Fil}^0 B_\mu\big)^{\varphi=1} = \mathbb{Q}_p$$

where the exponent "$\varphi{=}1$" means that we are taking the subspace of fixed points under φ. The reader should be aware that restricting to Fil^0 is essential: $B_\mu^{\varphi=1}$ is much bigger than $\mathbb{Q}_p$. Precisely, we have the so-called fundamental exact sequence:

$$0 \to \mathbb{Q}_p \to B_\mu^{\varphi=1} \to B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \to 0$$

where the map $B_\mu^{\varphi=1} \to B_{\mathrm{dR}}/B_{\mathrm{dR}}^+$ is induced by the natural inclusion $B_\mu \hookrightarrow B_{\mathrm{dR}}$.

4 Crystalline and de Rham representations

We keep the general notations of the previous section: the letter K denotes a finite extension of $\mathbb{Q}_p$, G_K is its Galois group, *etc.* So far, we have defined the periods rings B_{crys} and B_{dR} (together with some variants). By Fontaine's formalism (cf §1.4), these rings cut out full subcategories of $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. The objective of this section is to study these categories and to demonstrate that they are relevant for geometric purpose. We begin with a definition.

Definition 4.0.1. Let V be a finite dimension $\mathbb{Q}_p$-linear representation of G_K.

(i) We say that V is *crystalline* if it is B_{crys}-admissible.

(ii) We say that V is *de Rham* if it is B_{dR}-admissible.

Rephrasing the definition of B-admissibilty and using Theorems 3.4.1 and 3.4.2, we have:

$$\begin{aligned} V \text{ is crystalline} &\iff \dim_{K_0} \big(B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V\big)^{G_K} = \dim_{\mathbb{Q}_p} V, \\ V \text{ is de Rham} &\iff \dim_K \big(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V\big)^{G_K} = \dim_{\mathbb{Q}_p} V. \end{aligned}$$

Moreover, since B_{crys} is a subring of B_{dR}, any crystalline representation is de Rham.

4.1 Comparison theorems: statements

We start by discussing the geometric relevance of the notion of crystalline and de Rham representations. Our ambition is only to state the relevant theorems in this direction and definitely not to prove them. The most important ingredients of the proofs will be presented and discussed in Yamashita's lecture [43] and Andreatta and al.'s lecture [1] in this volume. From now on, we fix a proper smooth variety X defined over Spec K. (At least) two different cohomology theories taking coefficients in $\mathbb{Q}_p$ can be naturally attached to X, namely:

- the (algebraic) de Rham cohomology $H^\bullet_{\mathrm{dR}}(X)$ of X: each component $H^r_{\mathrm{dR}}(X)$ is a K-vector space endowed with a descreasing filtration, denoted by $\mathrm{Fil}^m H^r_{\mathrm{dR}}(X)$, with $\mathrm{Fil}^0 H^r_{\mathrm{dR}}(X) = H^r_{\mathrm{dR}}(X)$ and $\mathrm{Fil}^{r+1} H^r_{\mathrm{dR}}(X) = 0$
- the p-adic étale cohomology $H^\bullet_{\text{ét}}(X_{\bar K}, \mathbb{Q}_p)$ where $X_{\bar K} = \mathrm{Spec}\, \bar K \times_{\mathrm{Spec}\, K} X$: each component $H^r_{\text{ét}}(X_{\bar K}, \mathbb{Q}_p)$ is a $\mathbb{Q}_p$-vector space endowed with a continuous action of $\mathrm{Gal}(\bar K/K)$.

In the early 1970's, Grothendieck [26] wondered whether one can compare these cohomology groups. More precisely, he raised the so-called *problem of the mysterious functor*, asking for the existence of a purely algebraic recipe to recover $H^r_{\mathrm{dR}}(X)$ from $H^r_{\text{ét}}(X_{\bar K}, \mathbb{Q}_p)$. When $X_{\mathbb{C}}$ is a complex variety, the problem of the "mysterious" functor has been solved for a long time; indeed, the de Rham comparison theorem ensures that $H^r_{\mathrm{dR}}(X_{\mathbb{C}})$ is isomorphic to the singular cohomology of $X_{\mathbb{C}}(\mathbb{C})$ with coefficients in $\mathbb{C}$ (which plays the role of the étale cohomology). As we shall see, the p-adic case is more subtle.

Using standard arguments, one proves that $H^r_{\mathrm{dR}}(X)$ and $H^r_{\text{ét}}(X_{\bar K}, \mathbb{Q}_p)$ have the same dimension for all r. Thus $K \otimes_{\mathbb{Q}_p} H^r_{\text{ét}}(X_{\bar K}, \mathbb{Q}_p)$ has to be isomorphic to $H^r_{\mathrm{dR}}(X)$ as abstract K-vector spaces. However there does not exist any *functorial* isomorphism between them. Therefore the coincidence of dimensions cannot be considered as a satisyfing answer to Grothendieck's question.

Hodge-like decomposition theorems discussed in §1.2 (see in particular Eq. (5)) constitute a significant process towards Grothendieck's problem. Indeed they show, for some particular X's, that $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$ is isomorphic to the *graded* module of $H^r_{\text{dR}}(X)$ after extending scalars to $\mathbb{C}_p$. However the de Rham filtration on $H^r_{\text{dR}}(X)$ is not canonically split in the p-adic setting; therefore some information is lost when passing to the graduation. The point, which was first formulated by Fontaine and Jannsen, is that we can recover this missing information by extending scalars to the larger field B_{dR}. This is the content of the C_{dR} theorem[10]:

Theorem 4.1.1 (C_{dR}). *Let X be a proper smooth variety over* Spec K. *For all r, there exists a canonical isomorphism:*

$$\gamma_{\text{dR}}(X) : B_{\text{dR}} \otimes_K H^r_{\text{dR}}(X) \simeq B_{\text{dR}} \otimes_{\mathbb{Q}_p} H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p) \tag{36}$$

which respects filtrations and Galois action on both sides. Moreover $\gamma_{\text{dR}}(X)$ is functorial in X.

In the above theorem, the filtration on the source of $\gamma_{\text{dR}}(X)$ is the "convolution" filtration:

$$\text{Fil}^m\big(B_{\text{dR}} \otimes H^r_{\text{dR}}(X)\big) = \sum_{a+b=m} \text{Fil}^a B_{\text{dR}} \otimes_K \text{Fil}^b H^r_{\text{dR}}(X)$$

whereas, on the target, the filtration comes only from that on B_{dR}. In the same way, the Galois action on the source (resp. on the target) of (36) is the diagonal action (resp. the action coming from that on B_{dR}).

We observe that Theorem 4.1.1 implies readily that the $\mathbb{Q}_p$-linear representation $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$ is de Rham. Moreover, taking G_K-invariants on both side of (36), we find a natural isomorphism:

$$H^r_{\text{dR}}(X) \simeq \big(B_{\text{dR}} \otimes_{\mathbb{Q}_p} H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)\big)^{G_K} \tag{37}$$

which gives a satisfactory answer to Grothendieck's mysterious functor problem. Similarly, passing to the graduation in (36), we obtain the following Hodge-like decomposition:

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p) \simeq \bigoplus_{a+b=r} \mathbb{C}_p(\chi^{-a}_{\text{cycl}}) \otimes_K H^b(X, \mathcal{O}_X) \tag{38}$$

extending Tate's theorem on abelian varieties (*cf* §1.2). Observe in addition that the above isomorphism gives the Hodge–Tate decomposition of $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$. In particular, we see that all the Hodge–Tate weights of $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$ are in the range $[-r, 0]$.

The Fontaine–Mazur conjecture A classical application of Theorem 4.1.1 is to prove that a representation V does *not* come from geometry: if we can prove that V is not de Rham (or not Hodge–Tate), it can't arise as the étale cohomology of a proper smooth variety. One may ask for the converse: does any de Rham representation arise as a subquotient of a Tate twist of the étale cohomology of some variety? In the local situation considered up to now, the answer is negative. Nevertheless, a "global" variant of this question is conjectured to admit a positive answer. It is the so-called Fontaine–Mazur conjecture, which first appeared in [24].

Let F be a number field, that is a finite extension of $\mathbb{Q}$. For any prime ideal $\mathfrak{p}$ in $\mathcal{O}_F$ (the ring of integers of F), one can consider the field $F_\mathfrak{p}$ defined as the completion of F with respect to the $\mathfrak{p}$-adic topology. If p is the prime number defined by $p\mathbb{Z} = \mathbb{Z} \cap \mathfrak{p}$, the field $F_\mathfrak{p}$ is a finite extension of $\mathbb{Q}_p$. Moreover its absolute Galois group Gal$(\bar{\mathbb{Q}}_p/F_\mathfrak{p})$ embeds into Gal$(\bar{\mathbb{Q}}/F)$. This embedding is not unique but it is up to conjugacy by an element of Gal$(\bar{\mathbb{Q}}/F)$. Therefore, if V is

[10]This result is sometimes referred to as the C_{dR}-conjecture (even if it is now proved) since it has been a conjecture for a long time. The letter "C" in C_{dR} stands for "comparison" or "conjecture".

a $\mathbb{Q}_p$-representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$, its restrction to $\mathrm{Gal}(\bar{\mathbb{Q}}_p/F_\mathfrak{p})$ is well defined and it makes sense to wonder whether it is de Rham or not. In the same way, a representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ (with coefficients in any ring) is said to be *unramified* at $\mathfrak{p}$ if its restriction to $\mathrm{Gal}(\bar{\mathbb{Q}}_p/F_\mathfrak{p})$ is unramified (*i.e.* if the inertia subgroup of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/F_\mathfrak{p})$ acts trivially on it).

Conjecture 4.1.2 (Fontaine–Mazur). *We fix a number field F and a prime number p. Let V be a finite dimensional $\mathbb{Q}_p$-representation of* $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$. *We assume that:*

(i) for almost[11] *all prime ideals $\mathfrak{p} \in \mathcal{O}_F$, the representation V is unramified at $\mathfrak{p}$,*

*(ii) for all primes $\mathfrak{p}$ above p (*i.e. *such that $\mathbb{Z} \cap \mathfrak{p} = p\mathbb{Z}$), the representation $V_{|\mathrm{Gal}(\bar{\mathbb{Q}}_p/F_\mathfrak{p})}$ is de Rham.*

Then V appears as a subquotient of some $H^r_{\text{ét}}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)(\chi^m_{\text{cycl}})$ where r is a nonnegative integer, X is a proper smooth variety defined over Spec F *and m is an integer.*

When a representation V satisfies the conclusion of the above conjecture, we usually say that V *comes from geometry*. From the C_{dR}-theorem, we derive that every representation of the shape $H^r_{\text{ét}}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)(\chi^m_{\text{cycl}})$ comes from geometry. The Fontaine–Mazur conjecture then appears as a purely algebraic criterium to recognize representations coming from geometry among all representations.

We would like to emphasize that the Fontaine–Mazur conjecture might look surprising at first glance. Indeed it has been known for a long time that the Galois action on the étale cohomology satisfies many additional properties: for instance, the eigenvalues of the Frobenius acting on the étale cohomology have to take very particular values, known as Weyl numbers. However, these properties are not required in Fontaine–Mazur conjecture. It means that, assuming the conjecture to be true, they are implied by the unramified and the de Rham conditions, which is *a priori* rather unexpected.

Nowadays, the Fontaine–Mazur conjecture is still open. It was recently proved by Emerton [14] and Kisin [32] for two-dimensional representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ (satisfying some additional mild conditions) using the most recent developments in algebraic number theory (*e.g.* modularity lifting theorems, p-adic Langlands program). As far as we know, beyond the dimension 2, nothing is known.

The C_{crys}-theorem We now go back to the local setting and examine the case of the variety X has good reduction. We recall that this means that there exists a proper smooth variety $\mathcal{X}$ over Spec $\mathcal{O}_K$ whose generic fiber is X. We emphasize that the model $\mathcal{X}$ is required to be smooth; it is the crucial assumption.

When X has good reduction, the de Rham cohomology of X carries more structures. Indeed, assuming that X has good reduction, one can fix a model $\mathcal{X}$ as above and consider its special fiber $\bar{\mathcal{X}}$. It is a proper smooth scheme defined over Spec k. To $\bar{\mathcal{X}}$, one can attach a third cohomology group: its *crystalline* cohomology $H^r_{\text{crys}}(\bar{\mathcal{X}})$, defined by Berthelot [5]. We refer to [5, 8] for a complete exposition of the crystalline theory. For this article, let us just recall *very* briefly that, for all positive integer r, the crystalline cohomology $H^r_{\text{crys}}(\bar{\mathcal{X}})$ is a module over $W(k)$ endowed with an endomorphism $\varphi : H^r_{\text{crys}}(\bar{\mathcal{X}}) \to H^r_{\text{crys}}(\bar{\mathcal{X}})$ which is semi-linear with respect to the Frobenius on $W(k)$. In addition, the crystalline cohomology of $\bar{\mathcal{X}}$ is closely related to the de Rham cohomology of X through the Hyodo–Kato isomorphism $K \otimes_{W(k)} H^r_{\text{crys}}(\bar{\mathcal{X}}) \simeq H^r_{\text{dR}}(X)$. Putting $K_0 = W(k)[\frac{1}{p}]$ as before, we see that Hyodo–Kato isomorphism defines a K_0-structure in $H^r_{\text{dR}}(X)$, namely $K_0 \otimes_{W(k)} H^r_{\text{crys}}(\bar{\mathcal{X}})$. One can prove that this structure is canonical in the sense that it does not depend on the choice of a proper smooth model $\mathcal{X}$ of X.

Theorem 4.1.3 (C_{crys}). *Let X be a proper smooth variery over* Spec K *with good reduction. Let $\mathcal{X}$ denote a proper smooth model of X over* Spec $\mathcal{O}_K$. *For all r, there exists a canonical and functorial isomorphism:*

$$\gamma_{\text{crys}}(X) : B_{\text{crys}} \otimes_W H^r_{\text{crys}}(\bar{\mathcal{X}}) \simeq B_{\text{crys}} \otimes_{\mathbb{Q}_p} H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p) \qquad (39)$$

[11]"almost all" means "all expect possibly a finite number of them"

which respects Galois action and Frobenius action on both sides and such that $B_{\mathrm{dR}} \otimes \gamma_{\mathrm{crys}}(X)$ respects filtrations.

Here, the Frobenius action is defined on the source (resp. the target) of $\gamma_{\mathrm{crys}}(X)$ as the diagonal action (resp. the action given by $\varphi \otimes \mathrm{id}$). Theorem 4.1.3 shows that the representation $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$ is crystalline as soon as X has good reduction. (Remember that we already knew that this representation was de Rham thanks to the C_{dR}-theorem.) More precisely, taking G_K-invariants on both sides of (39), we obtain:

$$H^r_{\mathrm{crys}}(\mathcal{X}) \simeq \left(B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)\right)^{G_K} \tag{40}$$

which shows that, when X has good reduction, the étale cohomology of X not only determines its de Rham cohomology but also its canonical K_0-structure coming from the crystalline cohomology. After the results of §4.3 (to come up), it turns out that the converse also holds true: the crystalline cohomology, equipped with its Frobenius and the de Rham filtration after scalar extension to K, determines the étale cohomology.

A brief history of the C_{dR}-theorem Theorems 4.1.1 and 4.1.3 were first stated as conjecture by Fontaine and Jannsen just after Fontaine introduced the corresponding periods rings B_{dR} and B_{crys} respectively. Fontaine also designed a strategy to prove these conjectures. Very roughly, it can be summarized as follows:

1. prove the C_{crys}-conjecture;
2. extend the C_{crys}-conjecture to the semi-stable case[12];
3. derive the C_{dR}-conjecture by reduction to the semi-stable case.

The case of C_{crys} looks easier than that of C_{dR} because the isomorphism (39) can be understood as a kind of Kunneth formula. Indeed, the period ring B_{crys} has a nice cohomological interpretation, that is $B_{\mathrm{crys}} = H^0_{\mathrm{crys}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$. It then becomes plausible that $B_{\mathrm{crys}} \otimes_{K_0} H^r_{\mathrm{crys}}(\bar{\mathcal{X}})$ could have something to do with the cohomology of $X_{\bar{K}}$. Beyond this remark, it remained to find the way to go back and forth between the crystalline and the étale cohomologies. To this end, Fontaine and Messing proposed to use a third cohomology, the syntomic cohomology, and to compare it to both sides of the isomorphism (36). Using these ideas, they managed to prove the C_{crys}-theorem under the additional assumption that X has dimension at most $\frac{p-1}{2}$ [23].

Regarding the second step, Fontaine and Illusie introduced and proposed to develop log geometry. The main feature of log geometry is that it sees a normal crossing divisor as a log-smooth scheme. It then should be the right framework to perform local computations in the semi-stable case and then, hopefully, to extend the proof by Fontaine and Messing to all varieties admitting semi-stable reduction. The development of log geometry was achieved by the Japanese school [30, 27], who defined an analogue of the crystalline cohomology in this setting — the so-called log-crystalline cohomology — and related it to the de Rham cohomology *via* a log-analogue of the Hyodo–Kato isomorphism.

The initial idea for the third step was to prove that every proper smooth variety over Spec K admits semi-stable reduction after a finite extension. Unfortunately, this problem turns out to be quite difficult and is still open nowadays. Nevertheless, de Jong [28, 29, 6] proved a weaker result which was enough to complete the last step of Fontaine's strategy. In the very long paper [41], Tsuji gathered all these inputs and finally came up with a complete proof of the C_{dR}-theorem. The main ingredients of the proof will be presented in Yamashita's lecture [43, §2] in this volume.

In the meanwhile, Faltings published another proof of the C_{crys} and C_{dR}-theorem [16] (but did not state a semi-stable version). Faltings' strategy is quite different from Fontaine's one and

[12] We say that a variety X over K has *semi-stable* reduction if it has a proper model $\mathcal{X}$ over Spec $\mathcal{O}_K$ whose generic fibre is a divisor with normal crossings.

relies on *almost mathematics*, a theory specifically developped by Faltings for this application, which can be thought of as a wild generalization of Tate–Sen's methods presented in §2. The common idea which unifies these two proofs is, roughly speaking, to develop advanced methods to control extensions obtained by extracting p-th roots: in Fontaine's approach, it is achieved by the syntomic topology[13] whereas Faltings' initial idea is to work over infinite extensions obtained by extracting successive p-th roots and to use almost mathematics as the main tool to study the cohomology of varieties defined over such extensions.

More recently, Scholze designed a very powerful framework to do geometry over many "very ramified" bases including those obtained from usual $\mathbb{Z}_p$-schemes by adjoining iterated p-th roots: it is the theory of *perfectoid spaces* [36]. Based on this, he obtained in [37] a new proof of the C_{dR}-theorem which extends readily to *analytic* varieties (without any hypothesis of type Kähler)! This proof will be sketched in the article of Andreatta and al.'s in this volume [1].

4.2 More on de Rham representations

Now we have seen the relevance of crystalline and de Rham representations, it looks important to study systematically their properties. We start with the de Rham case. Let $\mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K)$ denote the category of $\mathbb{Q}_p$-linear de Rham representations of G_K. By Fontaine's general formalism, we know that $\mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K)$ is a full abelian subcategory of $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. It is moreover stable under direct sums, duals, tensor products, subobjects and quotients.

Theorem 4.2.1. *Any finite dimensional $\mathbb{C}_p$-admissible representation of G_K is de Rham.*

Proof. Let V be a finite dimensional $\mathbb{C}_p$-admissible representation of G_K. By Remark 2.2.7, there exists a finite extension L of K^{ur} such that V is $(L{\cdot}\hat{K}^{\mathrm{ur}})$-admissible. Since $L{\cdot}\hat{K}^{\mathrm{ur}} \subset B_{\mathrm{dR}}$, we conclude that V is de Rham. □

Another interesting result is that de Rham representations can be detected by looking at the restriction to open subgroups. Precisely, we have the following theorem.

Theorem 4.2.2. *Let L be a finite extension of K and let V be a finite dimensional $\mathbb{Q}_p$-linear representation of G_K. Then V is de Rham if and only if $V_{|G_L}$ is de Rham.*

Remark 4.2.3. In other terms, Theorem 4.2.2 says that the following diagram is cartesian.

$$\begin{array}{ccc}
\mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K) & \hookrightarrow & \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \\
{\scriptstyle \text{restriction}}\downarrow & & \downarrow{\scriptstyle \text{restriction}} \\
\mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_L) & \hookrightarrow & \mathrm{Rep}_{\mathbb{Q}_p}(G_L)
\end{array}$$

Proof of Theorem 4.2.2. By definition, if V is de Rham, the B_{dR}-semi-linear representation $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$ is trivial as a G_K-representation. It is then a fortiori trivial as a G_L-representation, which means that $V_{|G_L}$ is de Rham.

Conversely, let us assume that $V_{|G_L}$ is de Rham. Without loss of generality, we may assume that the extension L/K is Galois (if not, replace L by its Galois closure). Define $D = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_L}$, so that we have $\dim_L D = \dim_{\mathbb{Q}_p} V$. Moreover D inherits a semi-linear action of $\mathrm{Gal}(L/K)$. By Hilbert's theorem 90 (*cf* Theorem 1.3.3), D is spanned by a basis of fixed vectors. In other words, $\dim_K D^{\mathrm{Gal}(L/K)} = \dim_L D = \dim_{\mathbb{Q}_p} V$. Since $D^{\mathrm{Gal}(L/K)} = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$, we have proved that V is de Rham. □

[13] A morphism of schemes obtained by extraction of a p-th root of some function turns out to be a covering for the syntomic cohomology.

The function D_{dR}. If V is a de Rham representation of G_K, we define:

$$D_{\mathrm{dR}}(V) = \big(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V\big)^{G_K} = \mathrm{Hom}_{\mathbb{Q}_p[G_K]}\big(V^\star, B_{\mathrm{dR}}\big) \tag{41}$$

where $\mathrm{Hom}_{\mathbb{Q}_p[G_K]}$ refers to the set of $\mathbb{Q}_p$-linear G_K-equivariant morphisms and $V^\star$ is the dual representation of V. Fontaine's formalism shows that we have a canonical isomorphism:

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V \simeq B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V). \tag{42}$$

Remark 4.2.4. When V is the étale cohomology of a proper smooth variety X over Spec K, the isomorphism (42) is the isomorphism (36) of the C_{dR}-theorem. Notably, we have $H^r_{\mathrm{dR}}(X) = D_{\mathrm{dR}}(H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p))$ for all integer r.

Formula (41) defines a functor $D_{\mathrm{dR}} : \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K) \to \mathrm{Vect}_K$ where Vect_K is the category of finite dimensional vector spaces over K. One can actually be more precise and endow $D_{\mathrm{dR}}(V)$ with a filtration coming from the filtration on B_{dR}. Precisely, for an integer $m \in \mathbb{Z}$, we define:

$$\mathrm{Fil}^m D_{\mathrm{dR}}(V) = \big(\mathrm{Fil}^m B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V\big)^{G_K} = \mathrm{Hom}_{\mathbb{Q}_p[G_K]}\big(V^\star, \mathrm{Fil}^m B_{\mathrm{dR}}\big).$$

Clearly $\mathrm{Fil}^m D_{\mathrm{dR}}(V)$ is sub-K-vector space of $D_{\mathrm{dR}}(V)$ and $\mathrm{Fil}^{m+1} D_{\mathrm{dR}}(V) \subset \mathrm{Fil}^m D_{\mathrm{dR}}(V)$ for all m. Moreover observe that:

$$\begin{aligned} &\bigcap_{m\in\mathbb{Z}} \mathrm{Fil}^m D_{\mathrm{dR}}(V) = \mathrm{Hom}_{\mathbb{Q}_p[G_K]}\Big(V^\star, \textstyle\bigcap_{m\in Z} \mathrm{Fil}^m B_{\mathrm{dR}}\Big) = 0 \\ \text{and} \quad &\bigcup_{m\in\mathbb{Z}} \mathrm{Fil}^m D_{\mathrm{dR}}(V) = \mathrm{Hom}_{\mathbb{Q}_p[G_K]}\Big(V^\star, \textstyle\bigcup_{m\in Z} \mathrm{Fil}^m B_{\mathrm{dR}}\Big) = D_{\mathrm{dR}}(V), \end{aligned}$$

the second equality coming from the fact that $D_{\mathrm{dR}}(V)$ has finite dimension over K. Since again $D_{\mathrm{dR}}(V)$ is finite dimensional, we deduce that $\mathrm{Fil}^m D_{\mathrm{dR}}(V) = 0$ for $m \gg 0$ and $\mathrm{Fil}^m D_{\mathrm{dR}}(V) = D_{\mathrm{dR}}(V)$ for $m \ll 0$; we say that the filtration of $D_{\mathrm{dR}}(V)$ is *separated* and *exhaustive*.

With the above construction, we have promoted D_{dR} to a functor $D_{\mathrm{dR}} : \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K) \to \mathrm{MF}_K$ where MF_K denotes the category of finite dimension K-vector spaces equipped with a nonincreasing separated and exhaustive filtration by sub-K-vector spaces. This functor has an extra remarkable property given by the next proposition.

Proposition 4.2.5. *For any $V \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K)$ and any integer m, the isomorphism* (42) *identifies* $\mathrm{Fil}^m(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$ *with* $\mathrm{Fil}^m(B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V))$ *where, by definition:*

$$\begin{aligned} \mathrm{Fil}^m\big(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V\big) &= \mathrm{Fil}^m B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V \\ \textit{and} \quad \mathrm{Fil}^m\big(B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V)\big) &= \sum_{a+b=m} \mathrm{Fil}^a B_{\mathrm{dR}} \otimes_K \mathrm{Fil}^b D_{\mathrm{dR}}(V). \end{aligned}$$

Proof. The inclusion $\mathrm{Fil}^m(B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V)) \subset \mathrm{Fil}^m(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$ is easily checked. It is then enough to show that mapping:

$$f : \mathrm{gr}\big(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} D_{\mathrm{dR}}(V)\big) \longrightarrow \mathrm{gr}\big(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V\big) \simeq B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V$$

induced by the inverse of (42) is an isomorphism. For this, we consider the exact sequence $0 \to \mathrm{Fil}^{m+1} B_{\mathrm{dR}} \to \mathrm{Fil}^m B_{\mathrm{dR}} \to \mathbb{C}_p(\chi^m_{\mathrm{cycl}}) \to 0$. Tensoring it by V and taking the G_K-invariants, we obtain an injective morphism $h_m : \mathrm{gr}^m D_{\mathrm{dR}}(V) \hookrightarrow (\mathbb{C}_p(\chi^m_{\mathrm{cycl}}) \otimes_{\mathbb{Q}_p} V)^{G_K}$. Taking the direct sum of the h_m's, we end up with an injective K-linear mapping $h : \mathrm{gr}\, D_{\mathrm{dR}}(V) \hookrightarrow (B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V)^{G_K}$. Now observe that $\dim_K \mathrm{gr}\, D_{\mathrm{dR}}(V) = \dim_K D_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p} V$ since V is de Rham. On the other hand, $\dim_K (B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V)^{G_K} \leq \dim_{\mathbb{Q}_p} V$ by the general Fontaine's formalism. As a consequence,

h must be an isomorphism. We conclude the proof by remarking that $B_{\mathrm{HT}} \otimes h = f \circ g$ where g is the canonical mapping:

$$g : B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} \mathrm{gr}\, D_{\mathrm{dR}}(V) \longrightarrow \mathrm{gr}\big(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} D_{\mathrm{dR}}(V)\big).$$

By definition of the filtration on $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} D_{\mathrm{dR}}(V)$, g is surjective. Since h is a bijection, we deduce, first, that g is an isomorphism and, then, that f is an isomorphism as well. □

As a byproduct of the above proof, we obtain the following quite interesting corollary.

Corollary 4.2.6. *Let V de a de Rham representation of G_K. Then V is Hodge–Tate and its Hodge–Tate weights are the integers m for which $\mathrm{gr}^{-m} D_{\mathrm{dR}}(V) \neq 0$, the multiplicity of m being equal to $\dim_K \mathrm{gr}^{-m} D_{\mathrm{dR}}(V)$.*

Proof. The corollary follows from the isomorphism $B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V \simeq B_{\mathrm{HT}} \otimes_K \mathrm{gr}\, D_{\mathrm{dR}}(V)$, which was established in the proof of Proposition 4.2.5. □

For one dimensional representations, the converse of Corollary 4.2.6 holds. Indeed, if $\chi : G_K \to \mathbb{Q}_p^\times$ is a Hodge–Tate character, then there exists some integer m for which $\chi \cdot \chi_{\mathrm{cycl}}^m$ is $\mathbb{C}_p$-admissible. By Theorem 4.2.1, we deduce that $\chi \cdot \chi_{\mathrm{cycl}}^m$ is de Rham. Hence χ is de Rham as well. However for higher dimensional representation, there do exist Hodge–Tate representations which are not de Rham.

B_{dR}-representations. After what we have achieved so far, it is quite tempting to study B_{dR}-semi-linear representations on their own in the spirit of Sen's theory (presented in §2.3). This work was achieved by Fontaine in [22]. Let us give rapidly a few details on Fontaine's results. Let K_∞ denote the p-adic cyclotomic extension of K. Generalizing Sen's arguments, Fontaine first shows that any B_{dR}-semi-linear representation of G_K descends to $K_\infty(\!(t)\!)$. We are then reduced to study the $K_\infty(\!(t)\!)$-semi-linear representations of $\Gamma = \mathrm{Gal}(K_\infty/K)$. Fontaine then defines an analogue of the Sen's operator which is no longer a linear map, but instead a derivation. More precisely, given a $K_\infty(\!(t)\!)$-semi-linear representation W of Γ, Fontaine shows that, for $\gamma \in \Gamma$ sufficiently closed to the identity, the formula $\frac{\log \gamma}{\log \chi_{\mathrm{cycl}}(\gamma)}$ defines a K_∞-linear mapping $\nabla_W : W \to W$ which satisfies the Leibniz rule, *i.e.*

$$\nabla_W(fw) = \frac{df}{dt} \cdot w + f \cdot \nabla_W(w) \qquad (f \in K_\infty(\!(t)\!),\, w \in W).$$

Moreover, as in Sen's theory, this construction is functorial and the datum of ∇_W caracterizes the representation W. For much more details, we refer to Fontaine's original paper [22].

4.3 More on crystalline representations

Let $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{crys}}(G_K)$ be the category of $\mathbb{Q}_p$-linear crystalline representations of G_K. It is a abelian subcategory of $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_K)$, which is stable by direct sums, duals, tensor products, subobjects and quotients. Unlike the de Rham case, the fact that a representation is crystalline cannot be detected on the restriction to an open subgroup in general. Nevertheless, we have a weaker result in this direction.

Proposition 4.3.1. *Let V be a finite dimensional $\mathbb{Q}_p$-linear representation of G_K. Then:*

(i) if V is unramified (i.e. *the inertia subgroup acts trivially on V), then V is crystalline,*

(ii) if there exists a finite unramified *extension L of K such that $V_{|G_L}$ is crystalline, then V is crystalline.*

Proof. By Proposition 2.2.5, if V is unramified then it is $\hat{K}^{\mathrm{ur}}$-admissible. Since $\hat{K}^{\mathrm{ur}} \subset B_{\mathrm{crys}}$, it is then *a fortioti* crystalline. This proves (i).

We now assume that $V_{|G_L}$ is crystalline for some finite unramified extension L of K. Without loss of generality, we may assume that L/K is Galois. We let L_0 be the maximal unramified extension of $\mathbb{Q}_p$ inside L. Then $\mathrm{Gal}(L/K) \simeq \mathrm{Gal}(L_0/K_0)$. Set $D = (B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V)^{G_L}$; it is a L_0-vector space endowed with a semi-linear action of $\mathrm{Gal}(L/K) \simeq \mathrm{Gal}(L_0/K_0)$. By Hilbert's theorem 90, we have $\dim_{K_0} D^{\mathrm{Gal}(L/K)} = \dim_{L_0} D$. Moreover since $V_{|G_L}$ is crystalline, we know that $\dim_{L_0} D = \dim_{\mathbb{Q}_p} V$. Consequently $\dim_{K_0} D^{\mathrm{Gal}(L/K)} = \dim_{\mathbb{Q}_p} V$, which proves that V is crystalline because $D^{\mathrm{Gal}(L/K)} = (B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V)^{G_K}$. □

We insist again on the fact that the assumption that L/K is unramified is crucial in Proposition 4.3.1.(ii). For example, one can prove (using Proposition 4.3.2 below for example) that a character is crystalline if and only if it is the product of an unramified character by a power of the cyclotomic character. In particular the finite order character $\omega_{\mathrm{cycl}} = [\chi_{\mathrm{cycl}} \bmod p]$ of $G_{\mathbb{Q}_p}$ is not crystalline.

A finite dimensional $\mathbb{Q}_p$-linear representation that becomes crystalline over a finite extension (non necessarily ramified) is called *potentially crystalline*. Combining Theorems 2.2.1, 4.2.1, 4.2.2 and Proposition 4.3.1, we obtain the following diagram of implications:

$$\begin{array}{ccccccc}
\mathbb{C}_p\text{-admissible} & \Longrightarrow & \text{pot. crys.} & \Longrightarrow & \text{de Rham} & \Longrightarrow & \text{Hodge–Tate} \\
\Uparrow & & \Uparrow & & & & \\
\text{unramified} & \Longrightarrow & \text{crystalline} & & & &
\end{array}$$

Proposition 4.3.2. *A representation which is at the same time crystalline and $\mathbb{C}_p$-admissible is unramified.*

Remark 4.3.3. Recall that, for a Hodge–Tate representation, $\mathbb{C}_p$-admissibility means that all Hodge–Tate weights are 0. Proposition 4.3.2 then says that any crystalline representation with Hodge–Tate weights 0 is unramified.

Proof of Proposition 4.3.2. Let V be a crystalline $\mathbb{C}_p$-admissible representation. From Remark 2.2.7, we derive that there exists a finite extension L of K such that V is $(L{\cdot}\hat{K}_0^{\mathrm{ur}})$-admissible.

Let $f : V^\star \to B_{\mathrm{dR}}$ be a G_K-equivariant $\mathbb{Q}_p$-linear morphism. Since V is crystalline, we know that $f(V^\star) \subset B_{\mathrm{crys}}$. Similarly, using that V is $(L{\cdot}\hat{K}_0^{\mathrm{ur}})$-admissible, we find $f(V^\star) \subset (L{\cdot}\hat{K}_0^{\mathrm{ur}})$. On the other hand, we know that $L \otimes_{L_0} B_{\mathrm{crys}}$ embebs into B_{dR}. The canonical morphism $(L{\cdot}\hat{K}_0^{\mathrm{ur}}) \otimes_{\hat{K}_0^{\mathrm{ur}}} B_{\mathrm{crys}} \to B_{\mathrm{dR}}$ is then injective. As a consequence $(L{\cdot}\hat{K}_0^{\mathrm{ur}}) \cap B_{\mathrm{crys}} = \hat{K}_0^{\mathrm{ur}}$ and we deduce that f takes its values in $\hat{K}_0^{\mathrm{ur}}$. As a conclusion, $\mathrm{Hom}_{\mathbb{Q}_p[G_K]}(V^\star, B_{\mathrm{dR}}) = \mathrm{Hom}_{\mathbb{Q}_p[G_K]}(V^\star, \hat{K}_0^{\mathrm{ur}})$.

Since V is de Rham, we deduce from the above equality that V is $\hat{K}_0^{\mathrm{ur}}$-admissible. In particular V embeds into a direct sum of copies of $\hat{K}_0^{\mathrm{ur}}$. Since the inertia subgroup acts trivially on $\hat{K}_0^{\mathrm{ur}}$, it acts trivially on V as well. □

Example 4.3.4. We give an example of a two dimensional representation which is de Rham but not crystalline. For any positive integer n, let $\varepsilon_n \in \bar{K}$ be a primitive p^n-th root of unity. Similarly, let $\varpi_n \in \bar{K}$ be a p^n-root of p. For any $g \in G_{\mathbb{Q}_p}$, there exists a unique element $c(g) \in \mathbb{Z}_p$ such that $g\varpi_n = \varepsilon_n^{c(g)} \varpi_n$ for all n. In the language of §3, the previous equation reads:

$$g p^\flat = \underline{\varepsilon}^{c(g)} \cdot p^\flat \qquad (g \in G_{\mathbb{Q}_p}) \tag{43}$$

where $p^\flat = (p, \bar{p}_1, \bar{p}_2, \ldots)$ and $\underline{\varepsilon} = (1, \bar{\varepsilon}_1, \bar{\varepsilon}_2, \ldots)$ are the elements of $\mathcal{R}$ defined in §3.1. A direct computation shows that $c(gh) = c(g) + \chi_{\mathrm{cycl}}(g) \cdot c(h)$ (we say that c is a cocycle). From this observation, we deduce that the function:

$$G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{Q}_p), \qquad g \mapsto \begin{pmatrix} \chi(g) & c(g) \\ 0 & 1 \end{pmatrix}$$

is a group homomorphism and then defines a two dimensional $\mathbb{Q}_p$-linear representation V of $G_{\mathbb{Q}_p}$. We are going to compute the space $D = \mathrm{Hom}_{\mathbb{Q}_p[G_{\mathbb{Q}_p}]}(V, B_{\mathrm{dR}})$. By the general theory, we know that:

(i) D is a K-vector space of dimension at most 2,

(ii) $V^\star$ is de Rham if and only if $\dim_K D = 2$,

(iii) $V^\star$ is crystalline if and only if it is de Rham and any morphism in D falls in B_{crys}.

On the other hand, D is canonically in bijection with the set of pairs $(x, y) \in B_{\mathrm{dR}}^2$ such that:

$$gx = \chi_{\mathrm{cycl}}(g)x \quad \text{and} \quad gy = y + c(g)x \tag{44}$$

for all $g \in G_{\mathbb{Q}_p}$. The pair $(0, 1)$ is obviously a solution of (44). Taking Teichmüller representatives and then passing to the logarithm in (43), we find that $(t, \log[p^\flat])$ (where we recall that $t = \log[\underline{\varepsilon}]$) is formally another solution of (44). It remains to justify that $\log[p^\flat]$ makes sense in B_{dR}. To this end, we observe that it can be defined as follows:

$$\log[p^\flat] = \log \frac{[p^\flat]}{p} = -\sum_{i=1}^{\infty} \frac{1}{i} \cdot \left(1 - \frac{[p^\flat]}{p}\right)^i.$$

(here we have chosen the convention that $\log p = 0$). Note that the series converges in $\mathrm{Fil}^1 B_{\mathrm{dR}}^+$ because $1 - \frac{[p^\flat]}{p} \in \mathrm{Fil}^1 B_{\mathrm{dR}}^+$). The space D is two dimensional and spanned by $(0, 1)$ and $(t, \log[p^\flat])$. Hence $V^\star$ is de Rham. The fact that $\log[p^\flat] \notin B_{\mathrm{crys}}$, *i.e.* that $V^\star$ is not crystalline can be checked as follows. Assume by contradiction that $\log[p^\flat] \in B_{\mathrm{crys}}$. Then, it would lies in $\mathrm{Fil}^1 B_{\mathrm{crys}}$, so that $a = \frac{\log[p^\flat]}{t} \in \mathrm{Fil}^0 B_{\mathrm{crys}}$. Moreover, we would have $\varphi(a) = a$ since the Frobenius takes $[p^\flat]$ to $[p^\flat]^p$. By Proposition 3.4.4, this would implies that $a \in \mathbb{Q}_p$. Applying Galois to the relation $\log[p^\flat] = at$, we would obtain $a + c(g) = \chi_{\mathrm{cycl}}(g)$ for all $g \in G_K$, which is obviously not true. Finally, we deduce that $V^\star$ is not crystalline.

Remark 4.3.5. The representation V of the previous example is the prototype of *semi-stable* representations. On the geometric side, it corresponds to the Tate curve, which is the prototype of elliptic curve without good reduction. Semi-stable representations will be introduced and widely discussed in Brinon's lecture. In particular, it will be proved in [9, Proposition 2.7] is actually transcendantal over Frac B_{crys}.

About B_μ-admissibility. Recall that, in §3, we have introduced a whole family of rings B_μ's (where $\mu \geq 1$ is a real paramater); there rings serve as variants of B_{crys}, which have the advantage of exhibiting more pleasant properties from the algebraic and analytic point of view. The next theorem shows that changing B_{crys} by B_μ does not affect the notion of crystalline representation.

Theorem 4.3.6. *Let $\mu \geq 1$ and let V be a finite dimension $\mathbb{Q}_p$-linear representation of G_K. Then V is* crystalline *if and only if it is B_μ-admissible.*

Proof. Since the B_μ's form a decreasing sequence of rings and $B_\mu \subset B_{\mathrm{crys}} \subset B_{p-1}$ for each $\mu < p-1$, it is enough to show that B_μ-admissibility implies $B_{p\mu}$-admissibility for all $\mu \geq 1$. But the latter assertion follows from the fact that the Frobenius induces a Galois equivariant ring isomorphism $B_\mu \xrightarrow{\sim} B_{p\mu}$ and therefore an isomorphism $(B_\mu \otimes_{\mathbb{Q}_p} V)^{G_K} \simeq (B_{p\mu} \otimes_{\mathbb{Q}_p} V)^{G_K}$. □

The functor D_{crys}. When V is a crystalline representation of G_K, we set:

$$D_\mu(V) = \big(B_\mu \otimes_{\mathbb{Q}_p} V\big)^{G_K} = \mathrm{Hom}_{\mathbb{Q}_p[G_K]}\big(V^\star,\, B_\mu\big)$$

for $\mu \geq 1$, $\mu = \text{crys}$ or $\mu = \max$ (which is, we recall, a redundant notation for $\mu = 1$). By Theorem 4.3.6, $D_\mu(V)$ is a vector space over K_0 of dimension $\dim_{\mathbb{Q}_p} V$. Observe in addition that $D_\mu(V)$ is equipped with a Frobenius map $\varphi : D_\mu(V) \to D_\mu(V)$ which is semi-linear with respect to the Frobenius on K_0. Moreover, one checks easily that:

$$K \otimes_{K_0} D_\mu(V) = \big(B_{\mu,K} \otimes_{\mathbb{Q}_p} V\big)^{G_K} = \big(B_{\text{dR}} \otimes_{\mathbb{Q}_p} V\big)^{G_K} = D_{\text{dR}}(V).$$

Therefore $K \otimes_{K_0} D_\mu(V)$ comes equipped with a filtration, namely the de Rham filtration.

The inclusion $B_\mu \subset B_{\max}$ induces an injective K_0-linear mapping $f_\mu : D_\mu(V) \to D_{\max}(V)$, which commutes with all additional structures. Since the source and the target of f_μ are both K_0-vector spaces of dimension $\dim_{\mathbb{Q}_p} V$, we conclude that f_μ is an isomorphism. In other words, the functor D_μ does not depend on the choice of μ; in what follows, we will prefer the notation D_{crys} (in order to make apparent the fact that we are considering the crystalline case) but the reader should keep in mind that $D_{\text{crys}} = D_\mu$ for all μ.

The above constructions motivate the following definition.

Definition 4.3.7. A *filtered φ-module* over K is a K_0-vector space D equipped with a semi-linear endomorphism $\varphi : D \to D$ and a nonincreasing, exhaustive and separated filtration on $K \otimes_{K_0} D$.

We denote by $\text{MF}_K(\varphi)$ the category of filtered φ-modules over K (the morphisms are the K_0-linear mappings commuting with φ and preserving the filtration after scalar extension to K). We have a natural functor $\text{MF}_K(\varphi) \to \text{MF}_K$ taking D to $K \otimes_{K_0} D$ equipped with its filtration. Besides, the previous constructions give rise to a functor

$$D_{\text{crys}} : \text{Rep}^{\text{crys}}_{\mathbb{Q}_p}(G_K) \to \text{MF}_K(\varphi)$$

whose composite with $\text{MF}_K(\varphi) \to \text{MF}_K$ is D_{dR}.

Theorem 4.3.8. *The function D_{crys} is exact and fully faithful.*

Proof. The fact that D_{crys} is exact follows directly by a dimension argument.

Let $V \in \text{Rep}^{\text{crys}}_{\mathbb{Q}_p}(G_K)$ and set $D = D_{\text{crys}}(V)$. We then have a canonical isomorphism $B_{\text{crys}} \otimes_{\mathbb{Q}_p} V \simeq B_{\text{crys}} \otimes_{K_0} V$ which commutes with Frobenius and respects the filtration after extending scalars to B_{dR}. Taking the Fil^0 and the fixed points under the Frobenius and using Proposition 3.4.4, we obtain:

$$V = \big(B_{\text{crys}} \otimes_{K_0} D\big)^{\varphi=1} \cap \text{Fil}^0\big(B_{\text{dR}} \otimes_K D_K\big) \tag{45}$$

(where the supscript "φ=1" means that we are taking fixed points under the Frobenius). Formula (45) defines a functor $V_{\text{crys}} : \text{MF}_K(\varphi) \to \text{Rep}_{\mathbb{Q}_p}(G_K)$ and we have just proved that $V_{\text{crys}} \circ D_{\text{crys}}$ is the identity. This is enough to ensure that D_{crys} is fully faithful. □

Remark 4.3.9. In the proof of Theorem 4.3.8, instead of B_{dR}, we could have used the smaller ring $B_{\text{crys},K}$. Similarly, we could have replaced everywhere the subscript "crys" by μ for any real number $\mu \geq 1$.

Remark 4.3.10. Let A is an abelian variety over K with good reduction and let $A[p^\infty]$ be the p-divisible groups of its points of p^∞-torsion. The étale cohomology (resp. the crystalline cohomology) of A is then identified with the Tate module (resp. the Dieudonné module) of $A[p^\infty]$. The fact that D_{crys} is fully faithful then reflects the fact that Dieudonné modules (equipped with the de Rham filtration) classify p-divisible groups.

Admissibility for φ-modules. We say that a filtered φ-module over K is *admissible* if it belongs to the essential image of D_{crys}, and we denote by $\text{MF}^{\text{adm}}_K(\varphi)$ the full subcategory of $\text{MF}_K(\varphi)$ consisting of admissible filtered φ-modules. Theorem 4.3.8 indicates that D_{crys} induces an equivalence of categories $\text{Rep}^{\text{crys}}_{\mathbb{Q}_p}(G_K) \simeq \text{MF}^{\text{adm}}_K(\varphi)$. This result provides a very concrete description of crystalline representations as soon as we are able to recognize admissible filtered φ-modules among all filtered φ-modules.

This is actually possible: there exists an easy numerical criterium that caracterizes admissibility. We would like to conclude this article by stating it (without proof). Let $D \in \mathrm{MF}_K(\varphi)$ and set $d = \dim_{K_0} D$. The maximal exterior product $\det D = \bigwedge^d D$ has a natural structure of filtered φ-module: the Frobenius on it is $\bigwedge^d \varphi$ (where the latter φ is the Frobenius acting to D) and:

$$\mathrm{Fil}^m(K \otimes_{K_0} \det D) = \sum_{m_1+\ldots+m_d=m} \mathrm{Fil}^{m_1} D_K \wedge \mathrm{Fil}^{m_2} D_K \wedge \cdots \wedge \mathrm{Fil}^{m_d} D_K$$

where we have set $D_K = K \otimes_{K_0} D$. Since $\det D$ is one dimensional, there exists a unique integer m for which $\mathrm{Fil}^m(K \otimes_{K_0} \det D) = K \otimes_{K_0} \det D$ and $\mathrm{Fil}^{m+1}(K \otimes_{K_0} \det D) = 0$. This integer is called the *Hodge number* of D and is usually denoted by $t_H(D)$. It is an easy exercise to check that we have the following alternative formula for $t_H(D)$:

$$t_H(D) = \sum_{m \in \mathbb{Z}} m \cdot \dim_K \mathrm{gr}^m D_K.$$

Similarly, we can assign an integer to D measuring the action of the Frobenius. Precisely, if v is nonzero element of $\det D$, we have $\bigwedge^d \varphi(v) = \lambda v$ for some $\lambda \in K_0$. One checks easily that $v_p(\lambda)$ does not depend on the choice of V. We call it the *Newton number* of D and denote by $t_N(D)$.

Theorem 4.3.11. *A filtered φ-module D over K is admissible if and only if the two following conditions hold:*

(i) $t_H(D) = t_N(D)$

(ii) for all sub-K_0-vector space $D' \subset D$ stable by the Frobenius, we have $t_H(D') \leq t_N(D')$, where D' is endowed with the induced filtration defined by:

$$\mathrm{Fil}^m(K \otimes_{K_0} D') = (K \otimes_{K_0} D') \cap \mathrm{Fil}^m(K \otimes_{K_0} D) \qquad (m \in \mathbb{Z}).$$

Theorem 4.3.11 was first conjectured by Fontaine in [19]. It has been proved first by Colmez and Fontaine in [11] about twenty years later. Today, other proofs are been proposed by different authors [3, 31, 17], but Theorem 4.3.11 remains a difficult result in all cases. Kisin's proof [31] will be sketched in Brinon's lecture in this volume [9] (in the more general framework of filtered (φ, N)-modules).

Example 4.3.12. As an easy example, let us give a complete classification of filtered φ-modules of dimension 1. Let then $D \in \mathrm{MF}_K(\varphi)$ with $\dim_{K_0} D = 1$; write $D_K = K \otimes_{K_0} D$. Let e be a basis of D. Then, there exists $\lambda \in K_0$ such that $\varphi(e) = \lambda e$. Observe that if e is changed to ue (with $u \in K_0$), λ becomes $\lambda \cdot \frac{\varphi(u)}{u}$. By Hilbert's theorem 90, the elements of the form $\frac{\varphi(u)}{u}$ are exactly the elements of norm 1 over $\mathbb{Q}_p$. Therefore $N_{K_0/\mathbb{Q}_p}(\lambda)$ does not depend on a choice of e and is a complete invariant classying the possible φ's on D. Concerning the filtration, there exists a unique integer r such that $\mathrm{Fil}^m D_K = D_K$ if $m \leq r$ and $\mathrm{Fil}^m D_K = 0$ otherwise.

One sees immediately that the D is admissible if and only if $v_p(\lambda) = r$. Moreover, when admissibility holds, an easy computation shows that the attached Galois representation $V_{\mathrm{crys}}(D)$ is given by the character $\chi_{\mathrm{cycl}}^{-r} \cdot \mu_\alpha^{-1}$ with $\alpha = N_{K_0/\mathbb{Q}_p}(p^{-r}\lambda)$.

Example 4.3.13. We now investigate the admissible filtered φ-modules of dimension 2 over $\mathbb{Q}_p$. We then consider $D \in \mathrm{MF}_{\mathbb{Q}_p}^{\mathrm{adm}}(\varphi)$ with $\dim_{\mathbb{Q}_p} D = 2$. The filtration on D is easy to describe: there exist two integers r and s with $r \leq s$ and a line $L \subset D$ such that $\mathrm{Fil}^m D = D$ if $m \leq r$, $\mathrm{Fil}^m D = L$ if $r < m \leq s$ and $\mathrm{Fil}^m D = 0$ if $m > s$. If $r = s$, it follows from Proposition 4.3.2 that the crystalline representation associated to D (if D is admissible) has the form $V(\chi_{\mathrm{cycl}}^{-r})$ for an unramified representation V. We leave this case to the reader and assume now that $r < s$. Then L is uniquely determined.

We want to describe the action of the Frobenius $\varphi : D \to D$. Let us first notice that φ is a *linear* mapping because the Frobenius acts trivially on $\mathbb{Q}_p$. We first assume that L is stable by the Frobenius. We let $\alpha \in \mathbb{Q}_p$ be the scalar by which φ acts on L and we let β be the second eigenvalue of φ. From the admissibility condition, we deduce $v_p(\alpha) + v_p(\beta) = r + s$ and $v_p(\alpha) \geq s$. Therefore $v_p(\beta) \leq r < s$. Hence $\alpha \neq \beta$ and φ is diagonalizable. If L' denotes the eigenspace associated to β, we have $t_N(L') = v_p(\beta)$ and $t_H(L') = r$. By the admissibility condition, this implies that $v_p(\beta) = r$ and then $v_p(\alpha) = s$. Then L and L' are themselves *admissible* filtered φ-modules of dimension 1 and D splits as a direct sum $D = L \oplus L'$. The attached Galois representation is then a direct sum of two crystalline characters.

We now assume that L is not stable under φ. Let e_1 be a nonzero vector in L. Define $e_2 \in D$ by the equality $\varphi(e_1) = p^s e_2$. The family (e_1, e_2) is a basis of D in which the matrix of φ has the form:

$$\Phi = \begin{pmatrix} 0 & p^r a \\ p^s & p^r b \end{pmatrix}$$

for $a, b \in \mathbb{Q}_p$. The admissibility condition implies $v_p(\det \Phi) = r + s$, and then $a \in \mathbb{Z}_p^\times$. It also implies that any eigenvalue of Φ must have valuation at least r. But if $v_p(b) < 0$, we see on the Newton polygon of the characteristic polynomial of Φ, that Φ has an eigenvalue of valuation strictly less than r. Therefore, we conclude that $v_p(b) \geq 0$, *i.e.* $b \in \mathbb{Z}_p$. When $b \in \mathbb{Z}_p^\times$, φ has two eigenvalues of valuation r and s respectively. Let L_r and L_s be the corresponding eigenspaces. Since e_1 is not an eigenvector, we have $t_H(L_r) = t_H(L_s) = r$. Hence, L_r is admissible and we have the exact sequence $0 \to L_r \to D \to D/L_r \to 0$ is $\mathrm{MF}_{\mathbb{Q}_p}^{\mathrm{adm}}(\varphi)$. Passing to Galois representations, we find that $V_{\mathrm{crys}}(D)$ is a non split extension of $\mathbb{Q}_p(\chi_{\mathrm{cycl}}^{-s}\mu_\alpha)$ by $\mathbb{Q}_p(\chi_{\mathrm{cycl}}^{-r}\mu_\beta)$ with $\alpha, \beta \in \mathbb{Z}_p^\times$.

On the contrary, when $v_p(b) > 0$, D is admissible and irreducible in the category $\mathrm{MF}_{\mathbb{Q}_p}^{\mathrm{adm}}(\varphi)$. It then gives rise to an irreducible crystalline representation of dimension 2 of $G_{\mathbb{Q}_p}$, whose Hodge–Tate weights are r and s.

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