CAN WE DREAM OF A 1-ADIC LANGLANDS CORRESPONDENCE?

by

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Abstract. — After observing that some constructions and results in the p-adic Langlands programme are somehow independent from p, we formulate the hypothesis that this astonishing uniformity could be explained by a 1-adic Langlands correspondence.

To Catriona Byrne.

Contents

The Langlands programme is a far-reaching and influential web of theorems and conjectures which has motivated a lot of research in Number Theory and Arithmetic Geometry for more than fifty years. Very roughly, it stipulates a profound and meaningful correspondence between representations of Galois groups on the one hand and representations of reductive groups on the other hand. Many variations on this theme are actually possible, depending on which base field (number field, function field, p-adic field, *etc.*) and which category of coefficients we are working with.

The pioneer works in the Langlands programme were mostly concerned with $\mathbb{C}$-valued representations. However, since the beginning of the 21st century, a purely p-adic version of Langlands correspondence has emerged under the impulsion of Breuil. Nowadays, this p-adic correspondence is fully established for 2-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ but, beyond this, little is known. Many examples have however been worked out and several conjectures have been proposed—and sometimes proved—throughout the years. One of them is the Breuil-Mézard conjecture, which predicts that the geometrical properties of some Galois deformation spaces are directly related to the decomposition properties of some representations of the corresponding reductive group.

Looking more carefully into the aforementioned works, we notice that, in many cases, the underlying prime number p often plays a figurative role in the calculations. Typically, the relevant reductive groups are usually defined over $\mathbb{Z}$ and a significant part of the constructions and arguments can be carried out at this level. On the Galois side, this constancy is not so obvious but it is nevertheless visible; indeed, even though a prime number needs to be fixed from the very beginning, we often observe, at the end of the day, that the results of the computations are mostly independent from it.

We make the hypothesis that these strong properties of uniformity with respect to p could have a deep meaning and all be the consequences of a new type of Langlands correspondence, which should be considered as the common denominator of the p-adic Langlands correspondences when p varies. We call this new hypothetic correspondence the 1-*adic Langlands correspondence* because we believe that the natural language to formulate it is the mysterious theory of characteristic one whose main protagonist is the famous field with one element.

The aim of this note is to bring the reader to the agreement that our hypothesis is not crazy but has conceivable foundations and deserves consideration. We start our argumentation by reviewing in §1 some recent developments towards the Breuil-Mézard conjecture, with the objective to highlight the places where the arguments and/or the notions take a combinatorial flavour in which we have the feeling that the underlying prime number p plays a secondary role. Then, in §2, we briefly recall the philosophy of the field with one element and show that it is incredibly appropriate for interpreting many objects and carrying out many constructions encountered in §1. Finally, in order to give more substance to our dream, we conclude this article by an appendix in which we share some thoughts towards the development of a Galois theory in characteristic one, which is certainly a prerequisite for a 1-adic Langlands correspondence.

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1. Combinatorics around the Breuil-Mézard conjecture

Let $p > 2$ be a prime number. Throughout this section, we fix a finite extension K of $\mathbb{Q}_p$ and write $G_K = \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ for its absolute Galois group. The Breuil-Mézard conjecture is a concrete statement relating deformation spaces of representations of G_K, on the one hand, and representations of p-adic reductive groups, on the other hand. The aim of this section is, firstly, to recall the formulation of this conjecture and, secondly, to emphasize that, in many cases, it can be approched using combinatorial arguments and constructions. These observations will be the key to build bridges with the field with one element in §2.

1.1. Review on the Breuil-Mézard conjecture. — We denote by $\mathcal{O}_K$ (resp k_K) the ring of integers (resp. the residue field) of K. Let $\overline{\rho} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be a continuous $\overline{\mathbb{F}}_p$-representation of G_K of dimension n and let $R_{\overline{\rho}}$ denote the $\overline{\mathbb{Z}}_p$-algebra parametrizing the deformations of $\overline{\rho}$. In [**Ki3**], Kisin proved that $R_{\overline{\rho}}$ admits quotients with strong arithmetical interest. More precisely, given in addition the two following data:

- a *Hodge type* λ, that is, by definition, the datum of a tuple $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$ for all embeddings $\iota : k_K \hookrightarrow \overline{\mathbb{F}}_p$,
- an *inertial type* t, that is, by definition, a finite-dimensional $\overline{\mathbb{Q}}_p$-representation of the inertia subgroup $I_K \subset G_K$ having open kernel and admitting an extension to G_K,

Kisin constructed a surjective morphism of $\overline{\mathbb{Z}}_p$-algebras $R_{\overline{\rho}} \to R_{\overline{\rho}}^{\lambda,\mathrm{t}}$ that parametrizes the lifts of $\overline{\rho}$ which are potentially crystalline with Hodge-Tate weights $(\lambda_i + n - i)_{i,\iota}$ and inertial type t.

The Breuil-Mézard conjecture is a numerical relation between the Hilbert-Samuel multiplicity of the special fibre of $R_{\overline{\rho}}^{\lambda,\mathrm{t}}$, denoted by $e(R_{\overline{\rho}}^{\lambda,\mathrm{t}} \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p)$, and invariants coming from the representation theory of GL_n. Precisely, let L_λ be the irreducible algebraic $\overline{\mathbb{Z}}_p$-representation of GL_n of highest weight λ. After [**He, ScZi, CEG+**], we know that there is a finite dimensional smooth irreducible $\overline{\mathbb{Q}}_p$-representation $\sigma(\mathrm{t})$ of $\mathrm{GL}_n(\mathcal{O}_K)$ associated to t. We choose a $\mathrm{GL}_n(\mathcal{O}_K)$-stable $\overline{\mathbb{Z}}_p$-lattice L_t in $\sigma(\mathrm{t})$, form the tensor product $L_{\lambda,\mathrm{t}} = L_\mathrm{t} \otimes_{\overline{\mathbb{Z}}_p} L_\lambda$ and write its semi-simplification modulo p as follows:

$$\left(L_{\lambda,\mathrm{t}} \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p\right)^{\mathrm{ss}} \simeq \bigoplus_{\sigma \in \mathcal{D}} \sigma^{\oplus n_{\lambda,\mathrm{t}}(\sigma)}$$

where the sums runs over the set $\mathcal{D}$ of Serre weights σ, that is the set of (isomorphism classes of) irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_n(k_K)$.

***Conjecture 1* (Breuil-Mézard).** — *There exists a family of integers* $(\mu_{\overline{\rho}}(\sigma))_{\overline{\rho},\sigma}$, *called* intrinsic multiplicities, *such that the following numerical equality holds:*

$$e\left(R_{\overline{\rho}}^{\lambda,\mathrm{t}} \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p\right) = \sum_{\sigma \in \mathcal{D}} n_{\lambda,\mathrm{t}}(\sigma)\, \mu_{\overline{\rho}}(\sigma) \tag{1}$$

for all triples $(\overline{\rho}, \lambda, \mathrm{t})$ *as above.*

The Breuil-Mézard conjecture was first formulated for 2-dimensional representations in [**BrMé1**]. Since then, it has attracted a lot of attention. Kisin [**Ki1**] proved it when $K = \mathbb{Q}_p$ (and $n = 2$) by making use of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ and the (global) Taylor-Wiles-Kisin patching argument. Sander [**Sa**] and Paškūnas [**Pa**] gave a purely local alternative proof which has been extended later on by Hu and Tan to nonscalar split residual representations [**HuTa**]. For a general K, but still assuming $n = 2$, the conjecture was proved by Gee and Kisin [**GeKi**] (see also [**CEGS**, Appendix C]) when $\lambda = (0, 0)$ for each embedding (which corresponds to potentially Barsotti–Tate deformations).

The extension of the Breuil-Mézard conjecture to higher n came later. The case of 3-dimensional representations was considered and partially solved by Herzig, Le and Morra [**HLM**] and Le, Le Hung, Levin and Morra [**LLHLM1, LLHLM2**]. The formulation stated in dimension n in Conjecture 1 is due to Emerton and Gee [**EmGe**] (see also [**GHS**]).

1.2. Encoding representations with combinatorial data. — It turns out that, in some cases, all the objects that intervene in the statement of the Breuil-Mézard conjecture can be entirely described by combinatorial data. Besides, these explicit descriptions provide us with a new viewpoint on the conjecture, quite useful for attacking it.

The case of 2*-dimensional representations.* — We start with the case of GL_2 for which encodings are simpler and more is known. For simplicity, we assume further that $K/\mathbb{Q}_p$ is unramified, *i.e.* $K = \mathbb{Q}_{p^f}$ for some positive integer f. In dimension 2, the *irreducible* continuous representations $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ all take the form:

$$\overline{\rho} = \mathrm{Ind}_{G_{K'}}^{G_K} \left(\omega_{2f}^h \cdot \mathrm{nr}'(\theta)\right) \tag{2}$$

where K' is the unique unramified extension of degree 2 of K, ω_{2f} is the fundamental character of $G_{K'}$ of level $2f$ and $\mathrm{nr}'(\theta)$ denotes the unique unramified character of $G_{K'}$ sending the arithmetic Frobenius to θ. The parameters h and θ are an integer defined modulo $p^{2f}-1$ and an element of $\overline{\mathbb{F}}_p$ respectively.

Similarly, we have a complete description of *tame* inertial types, that are, by definition, those inertial types $\mathrm{t} : I_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ that factor through the tame inertia. Depending on whether they are reducible or not, they are of the form:

$$\mathrm{t} = \omega_f^\gamma \oplus \omega_f^{\gamma'} \quad \text{or} \quad \mathrm{t} = \mathrm{Ind}_{I_{K'}}^{I_K} \omega_{2f}^\gamma \tag{3}$$

with obvious notations. In the first case, we say that t has level f; otherwise, that it has level $2f$.

We now assume further that $\lambda = (0, 0)$ for each embedding. In this case, the integers $n_{\lambda,\mathrm{t}}(\sigma)$ are always 0 or 1 and we define $\mathcal{D}(\mathrm{t})$ as the set of Serre weights $\sigma \in \mathcal{D}$ for which $n_{\lambda,\mathrm{t}}(\sigma) = 1$. Similarly, it is conjectured that $\mu_{\overline{\rho}}(\sigma)$ is always 0 or 1 as well and we let $\mathcal{D}(\overline{\rho}) \subset \mathcal{D}$ be the locus over which $\mu_{\overline{\rho}}(\sigma)$ is strictly positive. Understanding the summation in the Breuil-Mézard conjecture (see Eq. (1)) then amounts to understanding the set $\mathcal{D}(\mathrm{t}, \overline{\rho}) = \mathcal{D}(\mathrm{t}) \cap \mathcal{D}(\overline{\rho})$.

It turns out that $\mathcal{D}(\mathrm{t})$ and $\mathcal{D}(\overline{\rho})$ admit very explicit combinatorial descriptions in terms of the parameters h, γ and γ' we introduced earlier. These descriptions first appeared in [**BDJ, BrPa**] and were then simplified in [**Da**]. Very roughly, once t (resp. $\overline{\rho}$) is fixed, the weights in $\mathcal{D}(\mathrm{t})$ (resp. in $\mathcal{D}(\overline{\rho})$) are parametrized by tuples $\underline{\varepsilon} = (\varepsilon_0, \ldots, \varepsilon_{f-1}) \in \{0,1\}^f$. Each such tuple produces a Serre weight by a simple recipe and the set $\mathcal{D}(\mathrm{t})$ (resp. $\mathcal{D}(\overline{\rho})$) is finally obtained by putting together all such weights. We underline that $\mathcal{D}(\mathrm{t})$ and $\mathcal{D}(\overline{\rho})$ have usually cardinality strictly less than 2^f because some $\underline{\varepsilon}$ may actually fail to produce a weight and it also happens that two different $\underline{\varepsilon}$ lead to the same weight.

The gene. — Building on the previous results, we gave in [**CDM3**] a purely combinatorial description of the intersection $\mathcal{D}(\mathrm{t}, \overline{\rho}) = \mathcal{D}(\mathrm{t}) \cap \mathcal{D}(\overline{\rho})$ in terms of the parameters h, γ and γ', assuming that the tame inertial type is reducible. More precisely, setting $q = p^f$ for simplicity, we considered the quantity $h - (q+1)\gamma' \bmod q^2 - 1$ and wrote its decomposition in radix p:

$$h - (q+1)\gamma' \equiv p^{2f-1}v_0 + p^{2f-2}v_1 + \cdots + pv_{2f-2} + v_{2f-1} \pmod{q^2 - 1}. \tag{4}$$

From the v_i's, we then formed a periodic sequence $\mathbb{X} = (X_i)_{i \in \mathbb{Z}}$ of period $2f$ assuming values in the finite set $\{\mathtt{A}, \mathtt{B}, \mathtt{AB}, \mathtt{O}\}$. The sequence $\mathbb{X}$ is called the *gene* of $(\mathrm{t}, \overline{\rho})$ and it satisfies the following rules (see [**CDM3**, Lemma B.1.3]):

- if $v_i = 0$ and $X_{i+1} = \mathtt{O}$, then $X_i = \mathtt{AB}$;
- if $v_i = 0$ and $X_{i+1} \neq \mathtt{O}$, then $X_i = \mathtt{A}$;
- if $v_i = 1$ and $X_{i+1} = \mathtt{O}$, then $X_i = \mathtt{O}$;
- if $v_i = 1$ and $X_{i+1} \neq \mathtt{O}$, then $X_i = \mathtt{B}$;
- if $v_i \geq 2$, then $X_i = \mathtt{O}$.

To each gene $\mathbb{X}$, we attached a set $\mathcal{W}(\mathbb{X})$ of *combinatorial weights*, which are sequences of length f with values in $\{0, 1\}$. We then proved (see [**CDM3**, Theorem 3.1.2]) that, if $\mathbb{X}$ denotes the gene of $(\mathrm{t}, \overline{\rho})$, there is a canonical bijection:

$$\mathcal{W}(\mathbb{X}) \xrightarrow{\sim} \mathcal{D}(\mathrm{t}, \overline{\rho}). \tag{5}$$

Beyond yielding an explicit description of $\mathcal{D}(\mathrm{t}, \overline{\rho})$ and opening concrete and algorithmical perspectives on the Breuil-Mézard conjecture, the above result raises new questions because it somehow shows that the dependence of $\mathcal{D}(\mathrm{t}, \overline{\rho})$ in t, $\overline{\rho}$ and even in the underlying prime number p itself, is very weak, given that the gene only retains little information about these data. In some sense, one can interpret the gene as the "skeleton" of the pair $(\mathrm{t}, \overline{\rho})$ that captures its most fundamental combinatorial properties in view of the Breuil-Mézard conjecture. In this perspective, the construction $\mathbb{X} \mapsto \mathcal{W}(\mathbb{X})$ should be thought of as the core factory of Serre weights, while the bijections (5), for varying t, $\overline{\rho}$ and p, appear as many tangible incarnations of this manufacture.

Higher dimension and group-theoretic formulation. — When we are moving to higher dimensions, the numerical descriptions we used previously cannot continue to be that simple but, interestingly, they have analogues which can be formulated in the language of group theory. In what follows, we continue to assume that $K = \mathbb{Q}_{p^f}$ for some positive integer f. In this setting, the relevant algebraic group is:

$$G = \left(\operatorname{Res}_{\mathcal{O}_K/\mathbb{Z}_p} \operatorname{GL}_{n/\mathcal{O}_K}\right) \times_{\mathbb{Z}_p} \overline{\mathbb{Z}}_p \simeq \prod_{\mathfrak{J}} \operatorname{GL}_{n/\overline{\mathbb{Z}}_p}$$

where $\mathfrak{J}$ is the set of embeddings $\mathcal{O}_K \hookrightarrow \overline{\mathbb{Z}}_p$ (or, equivalenty, $k_K \hookrightarrow \overline{\mathbb{F}}_p$) and will be identified with $\mathbb{Z}/f\mathbb{Z}$ in what follows. We denote by T the diagonal maximal torus of G. We let R be the set of roots of (G, T) and W be the corresponding Weyl group. The Borel $B \subset G$ of upper triangular matrices determines a subset $R^+ \subset R$ of positive roots. The group of characters $X^\star(T)$ can be canonically identified with $(\mathbb{Z}^n)^{\mathfrak{J}} = (\mathbb{Z}^n)^f$ and the Weyl group W is isomorphic to $\mathfrak{S}_n^f$. We shall also need the extended Weyl group $\widetilde{W}$ of G, defined by $\widetilde{W} = W \ltimes X^\star(T) \simeq (\mathfrak{S}_n \ltimes \mathbb{Z}^n)^f$.

In this setting, a Serre weight is an (isomorphism class of) irreducible $\overline{\mathbb{F}}_p$-representation of $\operatorname{GL}_n(\mathbb{F}_{p^f})$. It follows from a somehow classical argument of the theory of representations of reductive groups that Serre weights can be parametrized by certain characters of G. Precisely, after [**GHS**, Lemma 9.2.4], we know that if we set:

$$X_0(T) = \left\{\, \lambda \in X^\star(T) \text{ s.t. } \langle \lambda, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in R^+ \,\right\}$$
$$X_1(T) = \left\{\, \lambda \in X^\star(T) \text{ s.t. } 0 \leq \langle \lambda, \alpha^\vee \rangle \leq p-1 \text{ for all } \alpha \in R^+ \,\right\}$$

and let π denote the shift $(x_i)_{i\in\mathbb{Z}/f\mathbb{Z}} \mapsto (x_{i+1})_{i\in\mathbb{Z}/f\mathbb{Z}}$ on $X^\star(T) \simeq (\mathbb{Z}^n)^f$, there is a bijection:

$$F : X_1(T)/(p-\pi)X_0(T) \xrightarrow{\sim} \mathcal{D} \tag{6}$$

taking a character λ to the restriction to $\operatorname{Res}_{\mathcal{O}_K/\mathbb{Z}_p} \operatorname{GL}_{n/\mathcal{O}_K}(\mathbb{F}_p) = \operatorname{GL}_n(\mathbb{F}_{p^f})$ of the representation of $G(\mathbb{F}_p)$ induced by the algebraic representation of G of highest weight λ.

We also have a description of tame inertial types in terms of elements of the group $\widetilde{W}$ [**LMS, LLHLM1, BHHMS**]. Let $(s, \mu) \in \widetilde{W} = W \ltimes X^\star(T)$ and write $s = (s_0, \ldots, s_{f-1})$ with $s_i \in \mathfrak{S}_n$ for all i. Let r be the order of $s_0 s_{f-1} \cdots s_1 \in \mathfrak{S}_n$. We consider the unramified extension K' of K of degree r and let $k_{K'}$ denote its residue field. We let $\tilde{\omega}_{rf} : I_K = I_{K'} \to \overline{\mathbb{Z}}_p^\times$ be the Teichmüller lift of the Serre fundamental character of level rf. For $j \in \mathfrak{J}$, we put:

$$\eta_j = \begin{cases} (n-1, \ldots, 1, 0) & j\text{-th coordinate} \\ (0, \ldots, 0) & \text{elsewhere} \end{cases}$$

and $\eta = \sum_{j\in\mathfrak{J}} \eta_j$. We define $\alpha'_{(s,\mu)} \in X^\star(T)^{\operatorname{Hom}_{k_K\text{-alg}}(k_{K'}, \overline{\mathbb{F}}_p)} \simeq X^\star(T)^r \simeq (\mathbb{Z}^n)^{rf}$ by:

$$\alpha'_{(s,\mu),j} = s_1^{-1} s_2^{-1} \cdots s_j^{-1} (\mu_j + \eta_j)$$

and finally set:

$$\tau(s, \mu + \eta) = \bigoplus_{1\leq i\leq n} \tilde{\omega}_{rf}^{\sum_{j'=0}^{rf-1} \alpha'_{(s,\mu),j',i} p^{j'}}. \tag{7}$$

It is our tame inertial type.

Irreducible representations $\overline{\rho} : G_K \to \operatorname{GL}_n(\overline{\mathbb{F}}_p)$ can be encoded in a similar fashion. Moreover, when $n = 2$, it turns out that the explicit descriptions of $\mathcal{D}(\mathrm{t})$ and $\mathcal{D}(\overline{\rho})$ we have mentioned earlier can be rephrased in the language of group theory which was briefly sketched above (at least under sufficiently generic assumptions). We do not reproduce here the corresponding recipes (which involve the so-called *p-dot product*) but refer to Propositions 2.4.2 and 2.4.3 of [**BHHMS**] for more details. So far, we do not have any candidate for being a plausible replacement of the gene when $n > 2$. However, all the above constructions tend to show that, even though they now need to be formulated in the language of group theory, combinatorics is still here (and is maybe even more ubiquitous) in higher dimensions.

1.3. Explicit computations of $R_{\overline{\rho}}^{\lambda,\mathrm{t}}$. — The Breuil-Mézard conjecture is concerned with the special fibre of $R_{\overline{\rho}}^{\lambda,\mathrm{t}}$ but, of course, obtaining a complete description of the ring $R_{\overline{\rho}}^{\lambda,\mathrm{t}}$ has also interest for its own. For example, explicit presentations of some $R_{\overline{\rho}}^{\lambda,\mathrm{t}}$ have been used by Emerton, Gee and Savitt [**EGS**] to prove important conjectures stated by Breuil in [**Br**] about lattices in the cohomology of Shimura curves. In this subsection, we outline the standard strategy that is used to approach $R_{\overline{\rho}}^{\lambda,\mathrm{t}}$ and report on the results of some explicit computations.

Review on Kisin's construction of $R_{\overline{\rho}}^{\lambda,\mathrm{t}}$. — The main theoretical ingredient for studying deformations of $\overline{\rho}$ with prescribed Hodge type and inertial type is the theory of Breuil-Kisin, which provides a description of these deformations by means of semi-linear algebra. In our setting and assuming in addition that t is tame of level f and $K = \mathbb{Q}_{p^f}$ as we already did previously, a Breuil-Kisin module is a projective module $\mathfrak{M}$ over $\overline{\mathbb{Z}}_p \otimes_{\mathbb{Z}_p} \mathcal{O}_K[[u]] \simeq \overline{\mathbb{Z}}_p[[u]]^{\mathfrak{J}}$ equipped with two additional structures:

- a *Frobenius map* $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ which is semi-linear (with respect to the endomorphism of $\overline{\mathbb{Z}}_p \otimes_{\mathbb{Z}_p} \mathcal{O}_K[[u]]$ acting by the identity on $\overline{\mathbb{Z}}_p$, by the Frobenius of $\mathcal{O}_K$ and taking u to u^p) and satisfy additional properties,
- a *descent data*, that is a linear action of the group $\mathrm{Gal}(K[\sqrt[e]{p}]/K) \simeq \mathbb{Z}/e\mathbb{Z}$ (with $e = p^f - 1$) that commutes with the Frobenius.

A famous theorem of Kisin [**Ki2**] indicates that these modules are in correspondence with $\overline{\mathbb{Z}}_p$-representations of G_K that become crystalline over the extension $K[\sqrt[e]{p}]$. Besides, the Hodge type (resp. the inertial type) of the latter can be easily read off on the form of the Frobenius map (resp. of the descent data) on the former. Even better, the reduction modulo p of the representation associated to a Breuil-Kisin module $\mathfrak{M}$ is uniquely and entirely described by the module $\mathfrak{M} \otimes_{\mathcal{O}_K[[u]]} k_K((u))$ equipped with its additional structures.

The Breuil-Kisin theory then looks particularly well fitted for the study of the deformation rings $R_{\overline{\rho}}^{\lambda,\mathrm{t}}$ and it turns out that it indeed is. In [**Ki3**], Kisin constructed a scheme $\mathcal{GR}_{\overline{\rho}}^{\lambda,\mathrm{t}}$ parametrizing the Breuil-Kisin modules $\mathfrak{M}$ of Hodge type λ, inertial type t and having the additional property that $\mathfrak{M} \otimes_{\mathcal{O}_K[[u]]} k_K((u))$ corresponds to the given representation $\overline{\rho}$. This scheme is moreover equipped with a morphism $\mathcal{GR}_{\overline{\rho}}^{\lambda,\mathrm{t}} \to \mathrm{Spec}\, R_{\overline{\rho}}$ whose schematic image has closure $\mathrm{Spec}\, R_{\overline{\rho}}^{\lambda,\mathrm{t}}$. One should be careful however that the morphism:

$$\kappa : \mathcal{GR}_{\overline{\rho}}^{\lambda,\mathrm{t}} \to \mathrm{Spec}\, R_{\overline{\rho}}^{\lambda,\mathrm{t}} \tag{8}$$

is *not* an isomorphism in general because two p-torsion Breuil-Kisin modules may correspond to the same Galois representation. It is however always an isomorphism on the generic fibre. The special fibre of $\mathcal{GR}_{\overline{\rho}}^{\lambda,\mathrm{t}}$ is denoted by $\overline{\mathcal{GR}}_{\overline{\rho}}^{\lambda,\mathrm{t}}$ and is called the *Kisin variety*(1); in some sense, it measures the default for κ to be an isomorphism.

(1)The terminology "variety" is justified by the fact that the scheme $\overline{\mathcal{GR}}_{\overline{\rho}}^{\lambda,\mathrm{t}}$ is always of finite type over $\overline{\mathbb{F}}_p$.

Examples in dimension 2: the generic case. — The first examples of explicit calculations of certain rings $R_{\overline{\rho}}^{\lambda,\mathrm{t}}$ have been carried out by Breuil and Mézard in [**BrMé1**] and [**BrMé2**]. They considered the case where $\overline{\rho}$ is 2-dimensional and absolutely irreducible, $\lambda = (0,0)$ for each embedding and t is tame of level f. Under some additional assumptions of genericity on $\overline{\rho}$, they obtained, when $\mathcal{D}(\mathrm{t},\overline{\rho})$ is not empty:

$$R_{\overline{\rho}}^{\lambda,\mathrm{t}} \simeq \frac{\overline{\mathbb{Z}}_p[[X_i, Y_i, i \in \mathfrak{J}_{\mathrm{II}}, Z_j, j \in \mathfrak{J} \setminus \mathfrak{J}_{\mathrm{II}}]]}{(X_iY_i - p, i \in \mathfrak{J}_{\mathrm{II}})}. \tag{9}$$

for a certain subset $\mathfrak{J}_{\mathrm{II}}$ of $\mathfrak{J}$ (which depends on λ, t and $\overline{\rho}$). The aforementioned genericity assumptions play a quite important role in Breuil and Mézard's argument. In fact, they imply that the underlying Kisin variety is reduced to one point, which itself ensures that the morphism κ of Eq. (8) is an isomorphism. The computation of $R_{\overline{\rho}}^{\lambda,\mathrm{t}}$ then directly reduces to that of $\mathcal{GR}_{\overline{\rho}}^{\lambda,\mathrm{t}}$.

Before moving to nongeneric cases, it is important to comment on the subset $\mathfrak{J}_{\mathrm{II}}$ which appeared in Eq. (9). The triviality of the Kisin variety indicates that the module over $\overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} k_K((u)) \simeq \overline{\mathbb{F}}_p((u))^{\mathfrak{J}}$ corresponding to $\overline{\rho}$ contains a unique lattice $\mathfrak{M}(\overline{\rho})$ that is a Breuil-Kisin module of type (λ, t). Breuil and Mézard then defined the *shape* of $\mathfrak{M}(\overline{\rho})$: it is a finite sequence $(g_0, \ldots, g_{f-1})$ assuming values in the finite set $\{\mathrm{I}, \mathrm{II}\}$ which, roughly speaking, is obtained by looking at the form of the matrix of $\varphi_{\mathfrak{M}(\overline{\rho})}$ in bases diagonalizing the action of the descent data. The set $\mathfrak{J}_{\mathrm{II}}$ is then formed by the indices i for which g_i is II.

The shape also plays a key role on the GL_2-side of the Breuil-Mézard conjecture: in our setting, the cardinality of $\mathcal{D}(\mathrm{t},\overline{\rho})$ is $2^{\mathrm{Card}(\mathfrak{J}_{\mathrm{II}})}$ (which is the Hilbert-Samuel multiplicity of the special fibre of the ring $R_{\overline{\rho}}^{\lambda,\mathrm{t}}$ given by Eq. (9)) and, more precisely, we can explicitely parametrize the weights in $\mathcal{D}(\mathrm{t},\overline{\rho})$ by subsets of $\mathfrak{J}_{\mathrm{II}}$.

Examples in dimension 2: the nongeneric case. — Nongeneric cases are more complicated because they usually correspond to nontrivial Kisin varieties. In [**CDM2**], we computed these Kisin varieties when, as above, $\overline{\rho}$ is absolutely irreducible, $\lambda = (0,0)$ for each embedding and t is tame of level f. We recall that, in this setting, we have attached to the pair $(\mathrm{t},\overline{\rho})$ its gene $\mathbb{X}$ (see §1.2). Our results show that the Kisin variety is entirely determined by the gene. Being a little bit more precise, we showed that $\overline{\mathcal{GR}}_{\overline{\rho}}^{\lambda,\mathrm{t}}$ is a closed subscheme of $(\mathbb{P}^1_{\overline{\mathbb{F}}_p})^{\mathbb{Z}/f\mathbb{Z}}$ defined by equations of the form:

$$\lambda_i \, x_i \, y_{i+1} = \mu_i \, x_{i+1} \, y_i \tag{10}$$

where $[x_i : y_i]$ denotes the projective coordinates on the i-th copy of $\mathbb{P}^1_{\overline{\mathbb{F}}_p}$ and λ_i and μ_i are elements of $\{0,1\}$ that can be read off on the gene $\mathbb{X}$.

It is important to observe that the notion of shape can be extended to the nongeneric case as well. Indeed, each $\overline{\mathbb{F}}_p$-point x of $\overline{\mathcal{GR}}_{\overline{\rho}}^{\lambda,\mathrm{t}}$ corresponds, by definition, to a Breuil-Kisin module and so has a well defined shape $g(x) = (g_0(x), \ldots, g_{f-1}(x)) \in \{\mathrm{I}, \mathrm{II}\}^f$ in the sense of Breuil and Mézard. In full generality, the shape is thus no longer a unique element in $\{\mathrm{I}, \mathrm{II}\}^f$ but a function on the $\overline{\mathbb{F}}_p$-points of the Kisin variety taking values in $\{\mathrm{I}, \mathrm{II}\}^f$. We proved moreover that this function is lower-continuous (for the partial ordering on the codomain defined by $\mathrm{I} < \mathrm{II}$) and thus defines a stratification on $\overline{\mathcal{GR}}_{\overline{\rho}}^{\lambda,\mathrm{t}}$ by locally closed subschemes. As well as the Kisin variety, the shape stratification is entirely determined by the gene.

We then proposed the following conjecture.

Conjecture 2. —

(i) The generic fibre of $R_{\overline{\rho}}^{\lambda,\mathrm{t}}$ *is determined by the Kisin variety equipped with its shape stratification.*
(ii) The ring $R_{\overline{\rho}}^{\lambda,\mathrm{t}}$ *is determined by the gene.*

Regarding the first item of the conjecture, we were actually much more precise and exhibited a candidate for being the generic fibre of $R_{\overline{\rho}}^{\lambda,\mathrm{t}}$. Besides, our candidate is rather explicit: it is defined as the formal neighborhood of the Kisin variety in a certain blow-up of $(\mathbb{P}^1_{\overline{\mathbb{Z}}_p})^{\mathbb{Z}/f\mathbb{Z}}$. We refer to [**CDM2**, §5.4] for a complete exposition of this construction.

Conjecture 2 is known is most cases where the Kisin variety is trivial. It has also been checked in [**CDM1**] in one example where the Kisin variety is isomorphic to $\mathbb{P}^1_{\overline{\mathbb{F}}_p}$; in this case, the deformation ring we obtained is $\overline{\mathbb{Z}}_p[[X,Y,Z]]/(XY-p^2)$.

Higher dimension and group-theoretic formulation. — In dimension 3, tamely potentially crystalline deformation rings for small Hodge-Tate weights and generic Galois representations have been studied in [**LLHLM2**]. The authors also obtained explicit presentations of the deformation rings $R_{\overline{\rho}}^{\lambda,\mathrm{t}}$ in several cases. The equations they found are often quite similar to the one given in Eq. (9). The case of rank 2 unitary group has also been considered in [**KoMo**]. In this setting, it turns out that the explicit computations of the corresponding $R_{\overline{\rho}}^{\lambda,\mathrm{t}}$'s boil down to the case of GL_2 (with an additional polarization structure on the Breuil-Kisin modules). The final equations they get are then again similar to Eq. (9).

In all these situations, although computations are certainly much more difficult, it is important to underline that the basic ingredients are the same: we continue to have a Kisin variety, together with a shape stratification and we hope that these two objects strongly govern the final form of the deformation space. In full generality (*i.e.* for any reductive group, even not necessarily GL_n), the Kisin variety and the shape stratification are defined using techniques coming from group theory: the Kisin variety is a subscheme of the affine Grassmannian defined by an explicit condition, which can be formulated in terms of the Cartan decomposition, while the shape function $x \mapsto g(x)$ is defined by means of the Iwahori-Cartan decomposition (and it now takes values in the Iwahori-Weyl group). After [**CaLe**], we know moreover that Kisin varieties are related with Pappas-Zhu local models [**PaZh**]; in this connection, the shape stratification corresponds to the canonical stratification by affine Schubert varieties.

2. The field with one element

The theory of the field with one element ($\mathbb{F}_1$) starts with an observation of Tits [**Ti**] who notices inquiring numerical coincidences in the theory of algebraic linear groups. The most fundamental example is given by the group GL_n itself. Indeed, its number of points over the finite field $\mathbb{F}_q$ is given by:

$$\begin{aligned}\mathrm{Card}\,\mathrm{GL}_n(\mathbb{F}_q) &= (q^n-1)\cdot(q^n-q)\cdots(q^n-q^{n-1})\\ &= (q-1)^n\cdot q^{n(n-1)/2}\cdot[n]_q\cdot[n{-}1]_q\cdots[1]_q\end{aligned}$$

where $[i]_q = 1+q+\cdots+q^{i-1}$ is the q-analogue of i and letting q tends to 1, we find:

$$\mathrm{Card}\,\mathrm{GL}_n(\mathbb{F}_q) \sim_{q\to 1} (q-1)^n\cdot n!. \tag{11}$$

What is surprising is that $n!$ can be interpreted as the cardinality of the symmetric group $\mathfrak{S}_n$, which is nothing but the Weyl group of GL_n. Similar results hold more generally for a large family of groups, including the orthogonal groups, the sympletic groups and their scalar restrictions. After these observations, Tits asked if these numerical matchings could have deeper roots and proposed to build a geometry of the so-called *field with one element* with the objective to give a systematical and geometrical understanding of all the combinatorial structures and constructions which appear in the theory of Lie groups or algebraic groups.

Tits' vision was then popularized by Soulé who came up in [**So**] with a first tentative definition of affine varieties over $\mathbb{F}_1$. Later on, other constructions were proposed and the subject has attracted more and more attention for the last two decades [**De1, ToVa, CCM, CoCo, Co, LPL, Lo1, BBK**]; see [**Lo2**] for a recent review on this topic. Nowadays, the theory of $\mathbb{F}_1$ is not well estalished yet. However, several definitions for the category of schemes over $\mathbb{F}_1$ have been proposed over the years and significant progress towards Tits' initial dream have been realized. Besides, geometry over $\mathbb{F}_1$ has been extended to new contexts and has nowadays close interactions with Arakelov geometry and p-adic geometry. In particular, Bambozzi, Ben-Bassat and Kremnizer introduced in [**BBK**] analytic geometry over $\mathbb{F}_1$; they managed notably to construct a model of the Fargues-Fontaine curve [**FaFo**] in their theory. Up to our knowledge, this was the first connexion between the theory of Galois representations (incarnated here by p-adic Hodge theory) and the field with one element.

2.1. Clues in favor of a 1-adic Breuil-Mézard conjecture. — The main reason why we believe that a 1-adic version of the Breuil-Mézard conjecture is possible is that many objects involved in the formulation and/or the resolution of this conjecture do have natural seeds in $\mathbb{F}_1$-geometry. In this subsection, we list the most important of them and comment on their $\mathbb{F}_1$-aspects.

First of all, we notice that the Weyl group W and its extended version $\widetilde{W}$, which both play a quite important role in the construction of Serre's weights and inertial types, do have natural interpretations in characteristic one: the Weyl group is the set of $\mathbb{F}_1$-points of the underlying algebraic group (this property is Tits' dream, which is the main guide of the theory) while the extended Weyl group may be interpreted as its set of points over $\mathbb{F}_1(X)$ (see Eq. (12) in Appendix A.1) Moreover, the recipes used for constructing Serre's weights and inertial types from an element of $\widetilde{W}$ (see Eqs. (6) and (7)) are purely combinatorial and it is quite likely that they can be reformulated by means of $\mathbb{F}_1$-geometry.

The equations of the Kisin varieties we obtained in [**CDM2**] (see Eq. (10)) show that all of them are defined over $\mathbb{F}_1$. Similarly, the deformation spaces computed in [**BrMé1, BrMé2**] and [**LLHLM2**] all appear as product of discs and annuli (see particularly Eq. (9)); as a consequence, they all come throught scalar extensions from analytic spaces over $\mathbb{F}_1$ in the sense of [**BBK**]. Moreover, although the construction of blow-ups and formal neighborhoods was not addressed in [**BBK**], it looks quite plausible that the candidates for deformation spaces we introduced in [**CDM2**] are defined over $\mathbb{F}_1$ as well.

All the above examples show furthermore a strong uniform behaviour with respect to p. In the language of $\mathbb{F}_1$-geometry, this uniformity means that a whole family of Kisin varieties (resp. of deformation spaces) parametrized by p comes by scalar extension to $\mathbb{F}_p$ (resp. to $\mathbb{Q}_p$) from a *unique* variety (resp. analytic variety) over $\mathbb{F}_1$. This result might suggest the existence of a common denominator of the theory of Kisin varieties (resp. deformation spaces) which is defined in characteristic one and underpins some of their features we observe over the p-adics. This expectation is strengthened by the fact that Kisin varieties have a deep group-theoretic interpretation (see last paragraph of §1.3). The same remark is valid for the shape stratifications as well: they have good chance to be visible in characteristic one, given that thay are closely connected to affine Schubert varieties, which are themselves known to be defined over $\mathbb{F}_1$ [**LPL**].

The recipe of [**CDM3**], giving a combinatorial description of the set of common Serre's weights in terms of the corresponding gene, has also a strong $\mathbb{F}_1$-flavour. Concretely, what we expect is that:

(1) the gene is a sort of $\mathbb{F}_1$-encoding of the pair $(\mathrm{t}, \overline{\rho})$,
(2) the combinatorial weights of [**CDM3**] are the mirror of a notion of Serre's weights in characteristic 1,
(3) the association

$$\text{gene} \mapsto \text{set of combinatorial weights}$$

is the $\mathbb{F}_1$-incarnation of the construction $(\mathrm{t}, \overline{\rho}) \mapsto \mathcal{D}(\mathrm{t}, \overline{\rho})$.

Beyond the justifications coming from the constructions of [**CDM3**], we underline that there are other evidences supporting that Serre's weights in characteristic 1 should have something to do with combinatorial weights (which are, we recall, sequences of length f assuming values in $\{0, 1\}$). Indeed, mimicking the usual definition in characteristic p, we expect Serre's weights in characteristic 1 to be interpreted as $\bar{\mathbb{F}}_1$-representations (whatever it means) of the group $\mathrm{GL}_2(\mathbb{F}_{1^f})$. But, following Tits' vision, we can write:

$$\mathrm{GL}_2(\mathbb{F}_{1^f}) = \big(\mathrm{Res}_{\mathbb{F}_{1^f}/\mathbb{F}_1} \mathrm{GL}_2\big)(\mathbb{F}_1) = \mathrm{Weyl}\big(\mathrm{Res}_{\mathbb{F}_{p^f}/\mathbb{F}_p} \mathrm{GL}_2\big) = (\mathbb{Z}/2\mathbb{Z})^f$$

and we already see the set $\{0, 1\}$ entering into the scene. More precisely, we can define $\mathrm{Sym}^k \mathbb{F}_1^2$ as the set $\{X^k, XY^{k-1}, \ldots, Y^k\}$ and let $\mathrm{GL}_2(\mathbb{F}_1) = \mathbb{Z}/2\mathbb{Z}$ act on it by letting its unique nontrivial element operate by swapping X and Y (see Appendix A.1 for a justification of this definition). Similarly, if $\underline{k} = (k_0, \ldots, k_{f-1})$ is a tuple of integers, we let $\mathrm{Sym}^{\underline{k}} \mathbb{F}_1^2$ be the cartesian product of the $\mathrm{Sym}^{k_i} \mathbb{F}_1^2$ equipped with the induced action of $\mathrm{GL}_2(\mathbb{F}_{1^f}) = (\mathbb{Z}/2\mathbb{Z})^f$. It is then an easy exercise to check that $\mathrm{Sym}^{\underline{k}} \mathbb{F}_1^2$ is irreducible (*i.e.* the action is transitive) if and only if $k_i \in \{0, 1\}$ for all i.

We underline that Conjecture 2 is also in line with the above vision: roughly speaking, it stipulates that the mapping $(\mathrm{t}, \overline{\rho}) \mapsto R^{\mathrm{t}}_{\overline{\rho}}$ descends over $\mathbb{F}_1$.

Combining the previous observations and being quite optimistic, one might hope that the Pappas-Rapoport spaces [**PaRa**] and/or Emerton-Gee stacks [**EmGe**] themselves have a model over $\mathbb{F}_1$ and that the irreducible components of the special fibre of the latter will be related to the set of Serre's weights in characteristic 1.

2.2. Major challenges. — We do not hide that, if possible, devising a 1-adic Langlands correspondence (or a 1-adic Breuil-Mézard conjecture) will definitely not be a simple task. Actually, although the geometry over $\mathbb{F}_1$ has already been developed quite a lot, a lot of fundamental ingredients and objects of the Langlands programme are missing in characteristic 1.

To start with, we notice that extensions of $\mathbb{F}_1$, usually referred to as $\mathbb{F}_{1^n}$, have been already considered by several authors [**So, KaSm, Co**] but they have never been systematically studied. Moreover, in the above references, $\mathbb{F}_{1^n}$ is defined as the cyclotomic extension of $\mathbb{F}_1$ whose Galois group is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$, and not $\mathbb{Z}/n\mathbb{Z}$. This means in particular that we do not have apparently a nice analogue of the Frobenius endomorphism, which sounds annoying. An option for fixing this issue is to work with another version of $\mathbb{F}_{1^n}$ on which we impose by design the existence of a Frobenius of order n (see Appendix A.2 for a first rough proposal).

Similarly, the field of 1-adic numbers $\mathbb{Q}_1$ and its extensions have not attracted much attention so far. We mention however that Connes introduced the ring of Witt vectors over $\mathbb{F}_1$ in [**Co**] but we are afraid that Connes' treatement does not perfectly fit with our perspectives since it is eventually related to Banach algebras over the reals, and not over the p-adics. In Appendix A.3, using a (certainly too) naive definition of $\mathbb{Q}_1$, we start exploring its theory of finite extensions.

On a different note, it seems that the theory of representations of reductive groups over $\mathbb{F}_1$ have not been systematically studied yet. This could sound surprising given that representations of reductive groups over finite fields have been attracted a lot of attention for more than fifty years and the underlying theory includes a lot of combinatorial parts that have good chance to be "defined" over $\mathbb{F}_1$.

2.3. Conclusion. — Although that, clearly, many locks still need to be unlocked, we continue to believe that the 1-adic Langlands correspondence is possible. Besides, according to us, trying to develop it could be, at the same time, a wonderful motivation and guide for exploring the 1-adic world and for inspiring the p-adic Langlands correspondence by separating the universal combinatorial structures on the one hand and the more usual arithmetical properties on the other hand. We thus warmly encourage all contributions to this topic.

Appendix A

Remarks on Galois theory in characteristic 1

In this appendix, we start exploring the Galois properties of the field with one element $\mathbb{F}_1$ and the field of 1-adic numbers $\mathbb{Q}_1$. Our aim is not at all to elaborate a complete and coherent theory but to share our intuition and point out some difficulties. In §A.1, we review briefly the usual constructions of geometries over $\mathbb{F}_1$. We then successively address the Galois theory of $\mathbb{F}_1$ and $\mathbb{Q}_1$ in §A.2 and §A.3 respectively.

A.1. Brief review of geometry over $\mathbb{F}_1$. — Most of modern theories of geometry over $\mathbb{F}_1$ start with defining the category $\mathbb{F}_1$**-Vect** of $\mathbb{F}_1$-vector spaces. After Tits' observation that $\mathrm{GL}_d(\mathbb{F}_1)$ should be isomorphic to $\mathfrak{S}_d$, it is tempting to define a vector space over $\mathbb{F}_1$ simply as a set, this cardinality corresponding to its dimension over $\mathbb{F}_1$. By definition, a $\mathbb{F}_1$-linear morphism $V \to W$ is a set-theoretical *partially defined* function $f : V \to W$. The direct sum (resp. the tensor product) of two vector spaces is their disjoint union (resp. their cartesian product). Besides, with this point of view, the standard $\mathbb{F}_1$-vector space of dimension d, namely $\mathbb{F}_1^d$, is represented by the set $\{1, \ldots, d\}$; its group of automorphisms then coincides with $\mathfrak{S}_d$, *i.e.* $\mathrm{GL}_d(\mathbb{F}_1) = \mathfrak{S}^d$ as expected.

After vector spaces over $\mathbb{F}_1$, one introduces $\mathbb{F}_1$-algebras: they are, by definition, objects in commutative monoids in the category $\mathbb{F}_1$**-Vect**, *i.e.* sets equipped with a partially defined law of commutative monoids which is usually denoted with the multiplicative convention. Examples of $\mathbb{F}_1$-algebras include $X^{\mathbb{N}} = \{1, X, X^2, \ldots\}$ and $X^{\mathbb{Z}}$ which should be thought of as $\mathbb{F}_1[X]$ and $\mathbb{F}_1(X)$ respectively. Similarly, the $\mathbb{F}_1$-algebra $\mathbb{F}_1[X_1, \ldots, X_d]$ is realized by the monoid $X_1^{\mathbb{N}} X_2^{\mathbb{N}} \cdots X_d^{\mathbb{N}}$ whose elements are monomials in $X_1, \ldots, X_d$.

If M is a $\mathbb{F}_1$-algebra, it makes sense to define M-modules: they are sets endowed with an action of M. The standard free module of rank d over M is $M^{\oplus d} = \{1, \ldots, d\} \times M$ where M acts by multiplication on the second coordinate. It is an easy exercise to check that the group of M-linear automorphisms of $M^{\oplus d}$ is the semi-direct product $\mathfrak{S}_d \ltimes (M^{\mathrm{gp}})^d$ where M^{gp} denotes the subgroup of invertible elements of M. In particular, we get:

$$\mathrm{GL}_d\left(\mathbb{F}_1(X)\right) = \mathfrak{S}_d \ltimes \mathbb{Z}^d \tag{12}$$

and thus obtain, at least in the case of GL_d, a $\mathbb{F}_1$-style interpretation of the extended Weyl group.

Until this point, (almost) all theories agree on definitions but, when we are coming to $\mathbb{F}_1$-schemes, points of view start to diverge. Chronologically, the first approach is due to Deitmar [**De1**] and it closely follows the classical theory of schemes: Deitmar introduced spectra of monoids, equipped them with a topology and a notion of sheaves and finally glued them to get $\mathbb{F}_1$-schemes. Soon after, Toen and Vaquié [**ToVa**] developed the functorial point of view. Starting from the category $\mathbb{F}_1$-**Vect** (or, more generally, with an abstract monoidal symmetric category $\mathcal{C}$), they defined the category $\mathbb{F}_1$-**Alg** as we did before, introduced a notion of Zariski covering on it and finally defined $\mathbb{F}_1$-schemes as sheaves on $\mathbb{F}_1$-**Alg** for this Grothendieck topology. In [**Ve**], Vezzani proved (in a slightly different context) that Deitmar's construction on the one hand and Toen-Vaquié's approach on the other hand are equivalent, in the sense that they give rise to the same category of $\mathbb{F}_1$-schemes. Besides, both viewpoints include a functor of scalar extension:

$$\mathbb{F}_1\text{-}\mathbf{Sch} \to \mathbb{Z}\text{-}\mathbf{Sch}, \quad X \mapsto X_{\mathbb{Z}}$$

deriving from the construction $M \to \mathbb{Z}[M]$ at the level of monoids. Deitmar observed that toric varieties are defined over $\mathbb{F}_1$ but he also proved that there are essentialy the only ones [**De2**].

Another point of view on $\mathbb{F}_1$-schemes, which in some sense goes back to Soulé's original definition, was proposed in [**CoCo**] by Connes and Consani. They suggested to define a $\mathbb{F}_1$-scheme as a triple $(\tilde{A}, X_{\mathbb{Z}}, e_X)$ where $\tilde{A}$ is a $\mathbb{F}_1$-algebra, $X_{\mathbb{Z}}$ is a classical scheme and $e_X : (\operatorname{Spec} \tilde{A})_{\mathbb{Z}} \to X_{\mathbb{Z}}$ is a morphism of schemes inducing a bijection on k-points for any field k. In some sense, this approach separates the purely combinatorial part, which is encoded by $\tilde{A}$, and the geometrical part, which is delegated to the classical theory throught the scheme $X_{\mathbb{Z}}$. López Peña and Lorscheid [**LPL**] proved that this framework is more flexible in the sense that it allows for defining a much larger panel of varieties over $\mathbb{F}_1$; those include grassmannians, split reductive groups, Schubert varieties, *etc.* Besides, in a subsequent paper, Lorscheid [**Lo1**] concretized Tits' premonition by realizing the Weyl group of a split reductive group as its set of points over $\mathbb{F}_1$.

Beyond the field with one element, what we need for the purpose of this paper is the field of 1-adic numbers. Fortunately, this question has already been touched in the literature by several authors. In [**Co**], Connes came with a definition of Witt vectors over $\mathbb{F}_1$. However, Connes' construction looks a bit disconnected to our needs as it comes equipped with a scalar extension functor assuming values in Banach algebras over the reals, and not over the p-adics. In a different direction, Bambozzi, Ben-Bassat and Kremnizer [**BBK**] laid the foundations of the theory of analytic varieties over $\mathbb{F}_1$. Roughly speaking, their construction is similar to the ones we briefly sketched above except that, instead of starting with the category $\mathbb{F}_1$-**Vect**, they consider various categories of sets X equipped with a function $|\cdot| : X \to \mathbb{R}^+$ whose purpose is to model the norm map. They showed that balls and annuli of rigid geometry come from analytic varieties over $\mathbb{F}_1$; this is thus also the case for all the deformation spaces we encoutered in §1.

A.2. Galois theory over $\mathbb{F}_1$. — It is a natural expectation that $\mathbb{F}_1$ should have a finite extension of degree n for all n with Galois group isomorphic to $\mathbb{Z}/n\mathbb{Z}$. In the literature [**So, KaSm, Co**], this extension $\mathbb{F}_{1^n}$ is usually defined as the cyclotomic extension of $\mathbb{F}_1$, that is the $\mathbb{F}_1$-algebra represented by the monoid $X^{\mathbb{Z}/n\mathbb{Z}}$. However, it appears that this point of view is not perfectly suited to our purpose for at least two reasons:

(1) *The Galois theory is not the expected one.* Indeed, the group of automorphisms of the monoid $X^{\mathbb{Z}/n\mathbb{Z}}$ is $(\mathbb{Z}/n\mathbb{Z})^\times$ and not $\mathbb{Z}/n\mathbb{Z}$; in particular, we do not have a distinguished Frobenius endomorphism. This issue is maybe even more visible when we extend scalars to $\mathbb{Z}$ or $\mathbb{F}_p$. Indeed, with the above definition, one would get:
$$\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{F}_p = \mathbb{F}_p[X]/(X^n - 1)$$
which is certainly *not* $\mathbb{F}_{p^n}$ and which besides has a different Galois group.

(2) *The formation of $\mathbb{F}_{1^n}$-points does not behave as desired.* If X is a scheme defined over $\mathbb{F}_{1^n}$, it seems reasonable to expect that the set of $\mathbb{F}_{1^n}$-points of X agrees with the set of $\mathbb{F}_1$-points of $\mathrm{Res}_{\mathbb{F}_{1^n}/\mathbb{F}_1} X$. For $X = \mathrm{GL}_d$, this results in:
$$\mathrm{GL}_d(\mathbb{F}_{1^n}) = \mathrm{Weyl}\big(\mathrm{Res}_{\mathbb{F}_{p^n}/\mathbb{F}_p} \mathrm{GL}_d\big) = (\mathfrak{S}_d)^n$$
where p, here, denotes any auxiliary prime number. This expectation is reinforced by the fact that the group $(\mathfrak{S}_d)^n$ plays a quite important role in the Breuil-Mézard conjecture as recalled in §1. However, if one lets $\mathbb{F}_{1^n}$ be the $\mathbb{F}_1$-algebra corresponding to the monoid $X^{\mathbb{Z}/n\mathbb{Z}}$, one would obtain $\mathrm{GL}_d(\mathbb{F}_{1^n}) = \mathfrak{S}_d \ltimes (\mathbb{Z}/n\mathbb{Z})^d$ (see discussion before Eq. (12)) which is certainly *not* isomorphic to $(\mathfrak{S}_d)^n$ since even cardinalities differ!

The conclusion of these observations is that, although the cyclotomic extension of $\mathbb{F}_1$ is undoubtedly an interesting object, it is probably not the $\mathbb{F}_{1^n}$ we need for the applications we have in mind. Moreover, one checks that there is unfortunately no $\mathbb{F}_1$-algebra meeting all our requirements. Instead, we propose to define from scratch a theory of $\mathbb{F}_{1^n}$-vector spaces as we did previously for $\mathbb{F}_1$, trying as much as possible to incorporate the Frobenius action and keep its desired properties.

Definition A.1. — A $\mathbb{F}_{1^n}$*-vector space* is a set. Given two $\mathbb{F}_{1^n}$-vector spaces V and W, a $\mathbb{F}_{1^n}$*-linear morphism* $f : V \to W$ is the datum of n partially defined set-theoretical functions $f_1, \dots, f_n : V \to W$.

Again, the standard $\mathbb{F}_{1^n}$-vector space of dimension d is represented by the set $\{1, \dots, d\}$. We denote it by $(\mathbb{F}_{1^n})^d$, or simply $\mathbb{F}_{1^n}$ when $d = 1$, in what follows. It is obvious from the definition that the automorphism group of $(\mathbb{F}_{1^n})^d$ is $(\mathfrak{S}_d)^n$, *i.e.* we have the expected equality $\mathrm{GL}_d(\mathbb{F}_{1^n}) = (\mathfrak{S}_d)^n$. Similarly, extending the definition of $\mathbb{F}_1(X)$ to our new setting, one checks that $\mathrm{GL}_d(\mathbb{F}_{1^n}(X)) = (\mathfrak{S}_d \ltimes \mathbb{Z}^d)^n$.

Moreover, we have an obvious scalar extension functor $\mathbb{F}_1$-**Vect** $\to \mathbb{F}_{1^n}$-**Vect** acting on objects by $V \mapsto V$ and on morphisms by $f \mapsto (f, f, \dots, f)$. In what follows we shall use the notation $\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} V$ to denote the scalar extension of V from $\mathbb{F}_1$ to $\mathbb{F}_{1^n}$. Regarding scalar restriction, there are two different options to define it, namely:
$$\begin{array}{rrcl} \mathrm{aRes}_{\mathbb{F}_{1^n}/\mathbb{F}_1} : & \mathbb{F}_{1^n}\textbf{-Vect} & \longrightarrow & \mathbb{F}_1\textbf{-Vect} \\ & V & \mapsto & V^{\oplus n} \\ \mathrm{mRes}_{\mathbb{F}_{1^n}/\mathbb{F}_1} : & \mathbb{F}_{1^n}\textbf{-Vect} & \longrightarrow & \mathbb{F}_1\textbf{-Vect} \\ & V & \mapsto & V^{\otimes n}. \end{array}$$

(We recall that the direct sum and the tensor product over $\mathbb{F}_1$ are defined as the disjoint union and the cartesian product respectively.) The functor $\mathrm{aRes}_{\mathbb{F}_{1^n}/\mathbb{F}_1}$ (resp. $\mathrm{mRes}_{\mathbb{F}_{1^n}/\mathbb{F}_1}$) will be referred to as the *additive* (resp. the *multiplicative*) scalar restriction from $\mathbb{F}_{1^n}$ to

$\mathbb{F}_1$; hence the notation. Both versions look interesting given than they both appear as adjoints of the scalar extensions. Precisely, for $V \in \mathbb{F}_{1^n}$**-Vect** and $W \in \mathbb{F}_1$**-Vect**, we have:

$$\mathrm{Hom}_{\mathbb{F}_{1^n}\text{-}\mathbf{Vect}}(V,\, \mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} W) = \mathrm{Hom}_{F_1\text{-}\mathbf{Vect}}(\mathrm{aRes}_{\mathbb{F}_{1^n}/\mathbb{F}_1}(V),\, W),$$
$$\mathrm{Hom}_{\mathbb{F}_{1^n}\text{-}\mathbf{Vect}}(\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} W,\, V) = \mathrm{Hom}_{\mathbb{F}_1\text{-}\mathbf{Vect}}(W,\, \mathrm{mRes}_{\mathbb{F}_{1^n}/\mathbb{F}_1}(V)).$$

Besides, $\mathrm{aRes}_{\mathbb{F}_{1^n}/\mathbb{F}_1}(V)$ and $\mathrm{mRes}_{\mathbb{F}_{1^n}/\mathbb{F}_1}(V)$ are both equipped with a Frobenius which acts by permuting cyclically the summands/factors. More concretely, the Frobenius action on $\mathrm{aRes}_{\mathbb{F}_{1^n}/\mathbb{F}_1}(V) \simeq \mathbb{Z}/n\mathbb{Z} \times V$ is given by $(i, x) \mapsto (i{+}1, x)$ and it is given on $\mathrm{mRes}_{\mathbb{F}_{1^n}/\mathbb{F}_1}(V) \simeq V^n$ by $(x_1, \ldots, x_n) \mapsto (x_2, \ldots, x_n, x_1)$. We observe in particular that $\mathrm{aRes}_{\mathbb{F}_{1^n}/\mathbb{F}_1}(\mathbb{F}_{1^n}) \simeq \mathbb{Z}/n\mathbb{Z}$. In this sense, our definition meets the most standard presentation of $\mathbb{F}_{1^n}$ [**So, KaSm, Co**]; the main difference is that we do not retain the group structure on $\mathbb{Z}/n\mathbb{Z}$ but replace it by a Frobenius structure given by the shift. This slight modification in the point of view is actually enough to retrieve a cyclic Galois group of order n.

Proposition A.2. — *The group of automorphisms of* $\mathrm{aRes}_{\mathbb{F}_{1^n}/\mathbb{F}_1}(\mathbb{F}_{1^n})$ *commuting with the Frobenius action is the cyclic group of order n generated by the Frobenius.*

Proof. — An automorphism of $\mathrm{aRes}_{\mathbb{F}_{1^n}/\mathbb{F}_1}(\mathbb{F}_{1^n})$ is, by definition, a bijection $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. Requiring that it commutes with the Frobenius amounts to saying that $f(x+1) = f(x)+1$ for all $x \in \mathbb{Z}/n\mathbb{Z}$. Clearly, a function satisfying this condition must be of the form $x \mapsto x+a$, *i.e.* f is a power of the Frobenius. □

Proposition A.2 does not extend *verbatim* if we replace $\mathrm{aRes}_{\mathbb{F}_{1^n}/\mathbb{F}_1}$ by $\mathrm{mRes}_{\mathbb{F}_{1^n}/\mathbb{F}_1}$; indeed, given that $\mathrm{mRes}_{\mathbb{F}_{1^n}/\mathbb{F}_1}(\mathbb{F}_{1^n})$ is reduced to one point, its group of automorphisms is trivial as well. However, we can recover the expected group if we consider all objects $V \in \mathbb{F}_{1^n}$**-Vect** at the same time: the group of Frobenius-preserving automorphisms of the *functor* $\mathrm{mRes}_{\mathbb{F}_{1^n}/\mathbb{F}_1}$ is cyclic of order n and generated by the Frobenius.

Passing to the limit, we can similarly set up a theory of vector spaces over $\bar{\mathbb{F}}_1 = \varinjlim_n \mathbb{F}_{1^n}$.

Definition A.3. — A $\bar{\mathbb{F}}_1$*-vector space* is a set. Given two $\bar{\mathbb{F}}_1$-vector spaces V and W, a $\bar{\mathbb{F}}_1$*-linear morphism* $f : V \to W$ is the datum of a sequence $(f_i)_{i\geq 0}$ of partially defined functions from V to W such that for all $x \in V$, the sequence $(f_i(x))_{i\geq 0}$ is periodic.

As before, we have a scalar extension functor $\mathbb{F}_1$**-Vect** $\to \bar{\mathbb{F}}_1$**-Vect** which acts trivially on objects and takes a morphism f in $\mathbb{F}_1$**-Vect** to the constant sequence $(f, f, \ldots)$. More generally, there is a functor $\mathbb{F}_{1^n}$**-Vect** $\to \bar{\mathbb{F}}_1$**-Vect** mapping a $\mathbb{F}_{1^n}$-linear morphism $(f_1, \ldots, f_n)$ to the sequence $(f_{i \bmod n})_{i\geq 0}$. If V is finite dimensional over $\bar{\mathbb{F}}_1$ (*i.e.* if V is a finite set), any $\bar{\mathbb{F}}_1$-linear morphism with domain V comes from a $\mathbb{F}_{1^n}$-linear morphism for some n. Restrictions of scalars also exist in this context. Writing $\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$, they are given by:

$$\begin{array}{rrcl} \mathrm{aRes}_{\bar{\mathbb{F}}_1/\mathbb{F}_1} : & \bar{\mathbb{F}}_1\text{-}\mathbf{Vect} & \longrightarrow & \mathbb{F}_1\text{-}\mathbf{Vect} \\ & V & \mapsto & \hat{\mathbb{Z}} \times V \\ \mathrm{mRes}_{\bar{\mathbb{F}}_1/\mathbb{F}_1} : & \bar{\mathbb{F}}_1\text{-}\mathbf{Vect} & \longrightarrow & \mathbb{F}_1\text{-}\mathbf{Vect} \\ & V & \mapsto & \{\,\text{periodic sequences with values in } V\,\}. \end{array}$$

Furthermore, $\mathrm{aRes}_{\bar{\mathbb{F}}_1/\mathbb{F}_1}(V)$ and $\mathrm{mRes}_{\bar{\mathbb{F}}_1/\mathbb{F}_1}(V)$ are both equipped with a Frobenius endomorphism: on $\mathrm{aRes}_{\bar{\mathbb{F}}_1/\mathbb{F}_1}(V)$, it is $(i, x) \mapsto (i{+}1, x)$ while it acts by shifting the sequence by 1 on $\mathrm{mRes}_{\bar{\mathbb{F}}_1/\mathbb{F}_1}(V)$. One checks that $\mathrm{Aut}_\varphi(\mathrm{aRes}_{\bar{\mathbb{F}}_1/\mathbb{F}_1}(\bar{\mathbb{F}}_1)) \simeq \mathrm{Aut}_\varphi(\mathrm{aRes}_{\bar{\mathbb{F}}_1/\mathbb{F}_1}) \simeq \mathrm{Aut}_\varphi(\mathrm{mRes}_{\bar{\mathbb{F}}_1/\mathbb{F}_1}) \simeq \hat{\mathbb{Z}}$ where Aut_φ means the Frobenius-preserving automorphisms.

A.3. Galois theory over $\mathbb{Q}_1$. — In what follows, we view $\mathbb{Z}_1$ (resp. $\mathbb{Q}_1$) as the $\mathbb{F}_1$-analytic algebra (in the sense of [**BBK**]) corresponding to the monoid $\varpi^{\mathbb{N}}$ (resp. $\varpi^{\mathbb{Z}}$) endowed with the norm $\|\varpi^v\| = r^v$ where r is a fixed real number in $(0,1)$. Here ϖ is a formal notation for the uniformizer of $\mathbb{Z}_1$ and does not have further meaning. Our definition of $\mathbb{Z}_1$ might sound too naive as it seems to identify $\mathbb{Z}_1$ with $\mathbb{F}_1[[\varpi]]$; however, at least for the properties we want to illustrate in this appendix, making this confusion will not have undesirable consequences.

We now aim at defining several families of extensions of $\mathbb{Q}_1$ and studying their Galois properties. We start with unramified extensions: we set $\mathbb{Q}_{1^n} = \mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Q}_1$ for any positive integer n and $\mathbb{Q}_1^{\mathrm{ur}} = \bar{\mathbb{F}}_1 \otimes_{\mathbb{F}_1} \mathbb{Q}_1$. They are equipped with a Frobenius structure coming from the Frobenius on $\mathbb{F}_{1^n}$ (resp. $\bar{\mathbb{F}}_1$) and with a structure of $\mathbb{Q}_1$-algebra materialized by the multiplication morphism by ϖ at the level of monoids. One checks that the group of automorphisms of $\mathrm{aRes}_{\mathbb{F}_{1^n}/\mathbb{F}_1}(\mathbb{Q}_{1^n})$ (resp. of $\mathrm{aRes}_{\bar{\mathbb{F}}_1/\mathbb{F}_1}(\mathbb{Q}_1^{\mathrm{ur}})$) commuting with both structures) is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ (resp. to $\hat{\mathbb{Z}}$). We then get the expected Galois group.

We now come to the analogue of the tower of tamely ramified extensions. When p is an actual prime number, this tower is obtained by extracting e-th roots of the uniformizer for e coprime with p. There exists an obvious analogue of this construction over $\mathbb{Q}_1$: for any positive integer e (without any condition of coprimality), we consider the $\mathbb{F}_1$-analytic algebra $\mathbb{Q}_1[\sqrt[e]{\varpi}]$ defined by the underlying monoid $\varpi^{(1/e)\cdot\mathbb{Z}}$ equipped with the norm $\|\varpi^v\| = r^v$ $(v \in \frac{1}{e}\mathbb{Z})$. Clearly $\mathbb{Q}_1[\sqrt[e]{\varpi}]$ is an extension of $\mathbb{Q}_1$ and we can define generally $\mathbb{Q}_{1^n}[\sqrt[e]{\varpi}] = \mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Q}_1[\sqrt[e]{\varpi}]$ and $\mathbb{Q}_1^{\mathrm{ur}}[\sqrt[e]{\varpi}] = \bar{\mathbb{F}}_1 \otimes_{\mathbb{F}_1} \mathbb{Q}_1[\sqrt[e]{\varpi}]$. It is also possible to take the limit on e and define $\mathbb{Q}_1[\sqrt[\infty]{\varpi}]$ as the $\mathbb{F}_1$-analytic algebra associated to the monoid $\varpi^{\mathbb{Q}}$ with norm $\|\varpi^v\| = r^v$. We write $\mathbb{Q}_1^{\mathrm{tr}} = \bar{\mathbb{F}}_1 \otimes_{\mathbb{F}_1} \mathbb{Q}_1[\sqrt[\infty]{\varpi}]$; it is our candidate for being the maximal tamely ramified extension (or even an algebraic closure?) of $\mathbb{Q}_1$.

Devising a decent Galois theory for the extension $\mathbb{Q}_1^{\mathrm{tr}}/\mathbb{Q}_1^{\mathrm{ur}}$ looks more difficult. Indeed, given that a morphism of monoids $\mathbb{Q} \to \mathbb{Q}$ which acts by the identity on $\mathbb{Z}$ needs to be trivial, we conclude that there is no nontrivial morphism of $\mathbb{Q}_1^{\mathrm{ur}}$-algebras of $\mathbb{Q}_1^{\mathrm{tr}}$ in the sense of Definition A.1. In order to explain how this issue can be tackled, it will be more convenient to work with finite extensions. For any positive integer n, we define $K_n = \mathbb{F}_{1^n}[\sqrt[n]{\varpi}]$; it is the $\mathbb{F}_{1^n}$-algebra represented by the monoid $\eta^{\mathbb{Z}}$ where we have set $\eta = \sqrt[n]{\varpi}$ for simplicity. As we said earlier, there are no nontrivial automorphisms of $\mathbb{Q}_{1^n}$-algebras of K_n. The subtlety is that such automorphisms do exist after restricting scalars to $\mathbb{F}_1$. An explicit example is given by the morphism

$$\sigma_n : \mathrm{aRes}_{\mathbb{F}_{1^n}/\mathbb{F}_1}(K_n) \to \mathrm{aRes}_{\mathbb{F}_{1^n}/\mathbb{F}_1}(K_n)$$

corresponding to the map:

$$\begin{array}{rccc} \sigma_n^\sharp : & \mathbb{Z}/n\mathbb{Z} \times \eta^{\mathbb{Z}} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} \times \eta^{\mathbb{Z}} \\ & (i,\, \eta^j) & \mapsto & (i{+}j,\, \eta^j). \end{array}$$

One checks that $\sigma_n^\sharp$ is a morphism of monoids (where the factor $\mathbb{Z}/n\mathbb{Z}$ is endowed with its additive structure) acting by the identity on $\mathrm{aRes}_{\mathbb{F}_{1^n}/\mathbb{F}_1}(\mathbb{Q}_{1^n})$. Therefore, σ_n has all the virtues to be considered as an element of the Galois group $\mathrm{Gal}(K_n/\mathbb{Q}_{1^n})$, although it is not clear to us, here, why we need to retain the monoid structure on $\mathbb{Z}/n\mathbb{Z}$ whereas we argued earlier that it should be discarded in favor of the Frobenius structure. In any case, we have the following proposition.

Proposition A.4. — *The group of automorphisms of monoids of $\mathbb{Z}/n\mathbb{Z} \times \eta^{\mathbb{Z}}$ acting trivially on the submonoid $\mathbb{Z}/n\mathbb{Z} \times \varpi^{\mathbb{Z}}$ is cyclic of order n, generated by $\sigma_n^\sharp$.*

Proof. — Let f be an automorphism of $\mathbb{Z}/n\mathbb{Z} \times \eta^{\mathbb{Z}}$ satisfying the conditions of the proposition. By assumption f fixes $(1,1)$ and $(0,\varpi)$. Write $f(0,\eta) = (a, \eta^b)$ with $a \in \mathbb{Z}/n\mathbb{Z}$ and $b \in \mathbb{Z}$. Since f is a morphism of monoids, we must have $(na, \eta^{nb}) = (0, \varpi)$, showing that $b = 1$. Hence f takes the form $(i, \eta^j) \mapsto (i{+}aj,\, \eta^j)$ and the proposition follows. □

After what precedes, we are tempted to write:

$$\mathrm{Gal}(K_n/\mathbb{Q}_{1^n}) = \langle \sigma_n \rangle \simeq \mathbb{Z}/n\mathbb{Z}.$$

Moreover, noticing that σ_n commutes with the Frobenius, we conclude that:

$$\mathrm{Gal}(K_n/\mathbb{Q}_1) = \langle \varphi, \sigma_n \rangle \simeq (\mathbb{Z}/n\mathbb{Z})^2.$$

Passing to the limit, we would end up with $\mathrm{Gal}(\mathbb{Q}_1^{\mathrm{tr}}/\mathbb{Q}_1^{\mathrm{ur}}) \simeq \hat{\mathbb{Z}}$ and $\mathrm{Gal}(\mathbb{Q}_1^{\mathrm{tr}}/\mathbb{Q}_1) \simeq \hat{\mathbb{Z}}^2$.

In order to give more credit to this conclusion, we would like to make the comparison with the classical case of $\mathbb{Q}_p$ (where p is an actual prime number). For each positive integer n, we set $K_{p,n} = \mathbb{Q}_{p^n}[p^{1/(p^n-1)}]$; it is the maximal totally and tamely ramified Galois extension of $\mathbb{Q}_{p^n}$. As a consequence, the extensions $K_{p,n}$ are cofinal in the maximal tamely ramified extension of $\mathbb{Q}_p$. Besides, the Galois group of $K_{p,n}/\mathbb{Q}_p$ sits in the following exact sequence:

$$1 \longrightarrow \underset{\substack{\uparrow \\ \text{cyclic of} \\ \text{order } p^n-1}}{\mathrm{Gal}(K_{p,n}/\mathbb{Q}_{p^n})} \longrightarrow \mathrm{Gal}(K_{p,n}/\mathbb{Q}_p) \longrightarrow \underset{\substack{\uparrow \\ \text{cyclic of} \\ \text{order } n}}{\mathrm{Gal}(\mathbb{Q}_{p^n}/\mathbb{Q}_p)} \longrightarrow 1$$

which admits a section and provides a presentation of $\mathrm{Gal}(K_{p,n}/\mathbb{Q}_p)$ as a semi-direct product $\mathbb{Z}/n\mathbb{Z} \ltimes \mathbb{Z}/(p^n{-}1)\mathbb{Z}$ where $a \in \mathbb{Z}/n\mathbb{Z}$ acts on $\mathbb{Z}/(p^n{-}1)\mathbb{Z}$ by multiplication by p^a.

Setting naively $p = 1$ in what precedes, we find $p^n{-}1 = 0$ which does not really make sense since $\mathrm{Gal}(K_n/\mathbb{Q}_{1^n})$ cannot be decently of cardinality zero. Remembering what we did in Eq. (11), we instead write the factorization:

$$p^n - 1 = (p-1) \cdot (1 + p + \cdots + p^{n-1}) = (p-1) \cdot [n]_p.$$

At the level of groups, the above factorization reflects the fact that $\mathrm{Gal}(K_{p,n}/\mathbb{Q}_{p^n})$ admits a subgroup of order $[n]_p$, namely $\mathrm{Gal}(K_{p,n}/K_{p,1})$. When p goes to 1, the factor $[n]_p$ converges to n and then, passing to the limit, we expect the group $\mathrm{Gal}(K_n/K_1) = \mathrm{Gal}(K_n/\mathbb{Q}_{1^n})$ to be cyclic of order n, which is exactly what we have found earlier. Furthermore, when p tends to 1, the action of $\mathbb{Z}/n\mathbb{Z}$ on $\mathbb{Z}/(p^n{-}1)\mathbb{Z}$ (and consequently on all its subgroups) becomes trivial, confirming our prediction that $\mathrm{Gal}(K_n/\mathbb{Q}_1)$ should be a direct product of $\mathrm{Gal}(K_n/\mathbb{Q}_{1^n})$ by $\mathrm{Gal}(\mathbb{Q}_{1^n}/\mathbb{Q}_1)$.

Remark A.5. — There is however one small annoying point in what we have said: why is it legitimate to get rid of the factor $(p{-}1)$? If instead of discarding it without further discussion, we try to keep it, we come to the conclusion that there should be between $\mathbb{Q}_{1^n}$ and K_1 an extension of degree 0 or, say, of infinitesimal degree. This suggests that the extensions $\mathbb{Q}_{1^n}$ and K_1 need to be considered as different objects, which could be a way to explain that the prefactor $\mathbb{Z}/n\mathbb{Z}$ should be endowed with its Frobenius structure in the former case and with its monoid structure in the latter one (see discussion before Proposition A.4).

In the similar fashion that the factor $(q{-}1)^n$ in Eq. (11) corresponds to the n-dimensional torus of GL_n, it is tempting to interpret the Galois group of the ghost infinitesimal extension $K_1/\mathbb{Q}_{1^n}$ as the algebraic group $\mathbb{G}_m$ over $\mathbb{F}_1$. Similarly, it sounds plausible to interpret the cyclic group $\mathbb{Z}/n\mathbb{Z}$ (which is supposed to be the Galois group of K_n/K_1) as the group of $\mathbb{F}_1$-points of an algebraic group, maybe $\mathrm{aRes}_{\mathbb{F}_{1^n}/\mathbb{F}_1}(\mathbb{G}_a)$. All of this suggests that $\mathrm{Gal}(K_n/\mathbb{Q}_{1^n})$ could just be the pale reflection of an algebraic group $\mathbf{Gal}(K_n/\mathbb{Q}_{1^n})$ defined over $\mathbb{F}_1$ and sitting in an exact sequence of the form:

$$1 \to \mathrm{aRes}_{\mathbb{F}_{1^n}/\mathbb{F}_1}(\mathbb{G}_a) \to \mathbf{Gal}(K_n/\mathbb{Q}_{1^n}) \to \mathbb{G}_m \to 1.$$

And similarly, passing to the limit:

$$1 \to \mathrm{aRes}_{\bar{\mathbb{F}}_1/\mathbb{F}_1}(\mathbb{G}_a) \to \mathbf{Gal}(\mathbb{Q}_1^{\mathrm{tr}}/\mathbb{Q}_1^{\mathrm{ur}}) \to \mathbb{G}_m \to 1.$$

Beyond its own interest, this interpretation would provide us with a natural Frobenius structure (given by $i \mapsto i+1$) on $\mathrm{Gal}(\mathbb{Q}_1^{\mathrm{tr}}/\mathbb{Q}_1^{\mathrm{ur}}) \simeq \hat{\mathbb{Z}}$ after taking $\mathbb{F}_1$-points.

References

[BBK] Bambozzi F., Ben-Bassat O., Kremnizer K., *Analytic geometry over $\mathbb{F}_1$ and the Fargues-Fontaine curve*, Adv. in Math. **356**, 2019

[BDJ] Buzzard K., Diamond F., Jarvis F., *On Serre's conjecture for mod ℓ Galois representations over totally real fields,* Duke Math. J. **155**, 2010, 105–161

[BHHMS] Breuil C., Herzig F., Hu Y., Morra S., Schraen B., *Gelfand-Kirillov dimension and mod p cohomology for* GL_2, preprint, 2020, 101p

[Br] Breuil C., *Sur un problème de compatibilité local-global modulo p pour* GL_2, J. Reine Angew. Math. **692**, 2014, 1–76

[BrMé1] Breuil C., Mézard A., *Multiplicités modulaires et représentations de* $\mathrm{GL}_2(\mathbb{Z}_p)$ *et de* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ *en* $l = p$, Duke Math, J. **115**, 2002, no.2, 205-310

[BrMé2] Breuil C., Mézard A., *Multiplicités modulaires raffinées*, Bull. Soc. Math. France **142**, 2014, 127-175

[BrPa] Breuil C., Paškūnas V., *Towards a modulo p Langlands correspondence for* GL_2, Memoirs of A.M.S. **216**, 2012

[CCM] Connes A., Consani C., Marcolli M., *Fun with* $\mathbb{F}_1$, J. Number Theory **129**, 2009, 1532–1561

[CDM1] Caruso X., David A., Mézard A., *Calculs d'anneaux de déformations potentiellement Barsotti–Tate*, Transactions of the American Mathematical Society, American Mathematical Society **370** (9), 2018, 6041–6096

[CDM2] Caruso X., David A., Mézard A., *Variétés de Kisin stratifiées et déformations potentiellement Barsotti–Tate*, Journal of the Institute of Mathematics of Jussieu, Cambridge University Press (CUP) **15** (5), 2018, 1019–1064

[CDM3] Caruso X., David A., Mézard A., *Combinatorics of Serre weights in the potentially Barsotti-Tate setting*, 2021, 56p

[CaLe] Caraiani A., Levin B., *Kisin modules with descent data and parahoric local models*, Ann. Scient. ÉNS **51**, 2018, 181–213

[CEGS] Caraiani A., Emerton M., Gee T., Savitt D., *Moduli stacks of two-dimensional Galois representations*, Preprint 2019, 123p

[CEG+] Caraiani A., Emerton M., Gee T., Geraghty D., Paskunas V., Shin S.W., *Patching and the p-adic local Langlands correspondance*, Cambridge J. Math. **4.**, 2016, no.2, 197–287

[Co] Connes A., *The Witt construction in characteristic one and quantization* in Noncommutative geometry and global analysis **546**, 2011, Amer. Math. Soc., Providence, 83–113

[CoCo] Connes A., Consani C., *Schemes over* $\mathbb{F}_1$ *and zeta functions*, Compos. Math. **146**, 2010, 1383–1415

[Da] David A., *Poids de Serre dans la conjecture de Breuil–Mézard*, Contemporary Math. **691**, 2017, 133-156

[De1] Deitmar A., *Schemes over* $\mathbb{F}_1$ in *Number fields and function fields—two parallel worlds*, Progr. Math. **239**, 2005, Birkhäuser Boston, 87–100

[De2] Deitmar A., $\mathbb{F}_1$*-schemes and toric varieties*, Beiträge Algebra Geom. **49**, 2008, 517–525

[EmGe] Emerton M., Gee T., *A geometric perspective on the Breuil-Mézard conjecture*, J. Inst. Math. Jussieu **13**, 2014, no.1, 183-223

[EGS] Emerton M., Gee T., Savitt D., *Lattices in the cohomology of Shimura curves*, Invent. Math. **200**, 2015, no. 1, 1–96

[FaFo] Fargues L., Fontaine J.-M., *Courbes et fibrés vectoriels en théorie de Hodge p-adique*, Astérisque **406**, 2018, 382p

[GeKi] Gee T., Kisin M., *The Breuil-Mézard conjecture for potentially Barsotti–Tate representations*, Forum of Mathematics, Pi, 2014, 56p

[GHS] Gee T., Herzig F., Savitt D., *General Serre weights conjectures,* J. Eur. Math. Soc. (JEMS) **20** (2018), no.12, 2859–2949

[He] Henniart G., *Sur l'unicité des types pour* GL_2, Appendice of [**BrMé1**], Duke Math J. **115**, 2002, no.2, 298-310

[HLM] Herzig F., Le D., Morra S., *On mod p local-global compatibility for* GL_3 *in the ordinary case*, Compo. Math., 2017, 2215–2286

[HuTa] Hu Y., Tan F., *The Breuil-Mézard conjecture for non-scalar split residual representations*, Ann. Scient. ÉNS **48** (6), 2013

[KaSm] Kapranov M, Smirnov A., *Cohomology determinants and reciprocity laws: number field case*, unpublished letter, 1995

[Ki1] Kisin M., *The Fontaine-Mazur conjecture for* GL_2, J. Amer. Math. Soc. **22**, 2009, 641–690

[Ki2] Kisin M., *Moduli of finite flat group schemes and modularity*, Ann. of Math. **107**, 2009, 1085–1180

[Ki3] Kisin M., *Potentially semi-stable deformation rings*, J. Amer. Math. Soc. **21**, 2008, 513–546

[KoMo] Koziol K., Morra S., *Serre weight conjectures for p-adic unitary groups of rank* 2, to appear in Algebra & Number Theory, 2020, 79p

[LLHL] Le D., Le Hung B., Levin B., *Weight elimination in Serre-type conjectures,* Duke Math. J. **168** (2019), no.13, 2433–2506

[LLHLM1] Le D., Le Hung B., Levin B., Morra S., *Serre weights and Breuil's lattice conjecture in dimension three*, Forum of Math. Pi **8**, 2020, e5, 135p

[LLHLM2] Le D., Le Hung B., Levin B., Morra S., *Potentially crystalline deformation rings and Serre weight conjectures: shapes and shadows*, Inventiones Mathematica **212**, 1, (2018), 1-107

[LMS] Le D., Morra S., Schraen B., *Multiplicity one at full congruence level,* J. Inst. Math. Jussieu **21**, 2022, no.2, 637–658

[Lo1] Lorscheid O., *Algebraic groups over the field with one element*, Math. Z. **271**, 2012, 117–138.

[Lo2] Lorscheid O., $\mathbb{F}_1$ *for everyone*, Jahresber. Dtsch. Math.-Ver. **120**, 2018, 83–116.

[LPL] López Peña J., Lorscheid O., *Torified varieties and their geometries over* $\mathbb{F}_1$, Math. Z. **267**, 2011, 605–643.

[Pa] Paškūnas V., *On the Breuil-Mézard conjecture*, Duke Math. J. **164** (2), 2015, 297-359.

[PaRa] Pappas G., Rapoport M., φ*-modules and coefficient spaces*, Mosc. Math. J. **9**, 2009, 625–663

[PaZh] Pappas G., Zhu X., *Local models of Shimura varieties and a conjecture of Kottwitz*, Invent. Math. **194**, 2013, 147–254

[Sa] Sander F., *A local proof of the Breuil-Mézard conjecture in the scalar semi-simplification case*, J. Lond. Math. Soc. **94**(2), 2016, 447–461.

[ScZi] Schneider P., Zink E.-W. *K-types for the tempered components of a p-adic general linear group,* J. Reine Angew. Math. **517** (1999), 161–208, with an appendix by Schneider P. and Stuhler U.

[So] Soulé C. *Les variétés sur le corps à un élément*, Mosc. Math. J., **4**, 2004, 217–244.

[Ti] Tits J. *Sur les analogues algébriques des groupes semi-simples complexes* in Colloque d'algèbre supérieure, tenu à Bruxelles du 19 au 22 décembre 1956, Centre Belge de Recherches Mathématiques, Établissements Ceuterick, Louvain, 1957, 261–289.

[ToVa] Toën B., Vaquié M., *Au-dessous de* Spec $\mathbb{Z}$, J. K-Theory **3**, 2009, 437–500

[Ve] Vezzani A., *Deitmar's Versus Toën-Vaquié's schemes over* $\mathbb{F}_1$, Math. Z. **271**, 2012, 911–926.

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