COMPUTATION OF CLASSICAL AND v-ADIC L-SERIES OF t-MOTIVES

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Abstract. We design an algorithm for computing the L-series associated to an Anderson t-motives, exhibiting quasilinear complexity with respect to the target precision. Based on experiments, we conjecture that the order of vanishing at $T = 1$ of the v-adic L-series of a given Anderson t-motive with good reduction does not depend on the finite place v.

Contents

Let $\mathbb{F}$ be a finite field with q elements. *Arithmetic in function fields* is a branch of arithmetic geometry which consists in replacing any occurrence of the ring $\mathbb{Z}$ by the polynomial ring $\mathbb{F}[t]$ and to study its "number theory". Compared to classical arithmetic, this has the benefit of being generally easier to handle. One may play along the existence of a prime field, that of the Frobenius endomorphism and the quite convenient fact that the theory at the infinite place is non-archimedean.

Arithmetic of function fields went so far to offer a remarkably simple analogue of the still conjectural category of mixed motives over $\mathbb{Q}$. Objects of this category–called *Anderson t-motives over* $\mathbb{F}(\theta)$– consist of finite free modules M over $\mathbb{F}(\theta)[t]$ equipped with a Frobenius-semilinear isomorphism τ_M of M with poles along the diagonal ideal $(t-\theta)$. One may attach various realizations to a t-motives, including an *ℓ-adic realization* $\mathrm{T}_\ell\, \underline{M}$: here $\ell \in \mathbb{F}[t]$ is a monic irreducible polynomial, and $\mathrm{T}_\ell\, \underline{M}$ is a finite free $\mathbb{F}[t]^\wedge_\ell$-module equipped with a continuous action of the absolute Galois group $G_{\mathbb{F}(\theta)}$ of $\mathbb{F}(\theta)$.

Anderson t-motives also possess function fields valued L-series which are defined under no conjecture and which shares striking similarities with motivic L-functions. They are formally defined as the product

$$L_S(\underline{M},T) = \prod_{\mathfrak{p}} \det_{\mathbb{F}[t]^\wedge_\ell} \left(\mathrm{id} - T^{\deg \mathfrak{p}} \mathrm{Frob}_{\mathfrak{p}}^{-1} | (\mathrm{T}_\ell\, \underline{M})^{I_{\mathfrak{p}}}\right)^{-1} \in \mathbb{F}(t)[[T]] \quad (1)$$

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which ranges over the monic irreducible polynomials in $\mathbb{F}[t]$; in each factor, ℓ is chosen distinct from $\mathfrak{p}$. The group $I_\mathfrak{p} \subset G_{\mathbb{F}(\theta)}$ denotes the inertia group at $\mathfrak{p}$ and $\mathrm{Frob}_\mathfrak{p} \in G_{\mathbb{F}(\theta)}/I_\mathfrak{p}$ denotes the arithmetic Frobenius. To make sense of formula (1), one should check first that the principal term of the product is a polynomial in $\mathbb{F}[t][T]$ independent from ℓ. This is done below in Corollary 1.2.6.

Given a finite place v of $\mathbb{F}[t]$, we also define a series $L_v(\underline{M}, T)$ by a similar product (1) but removing the local factor at $\mathfrak{p} = v$. In what follows, we also write $L_\infty(\underline{M}, T)$ for $L(\underline{M}, T)$.

Remark. Similar L-series have already been introduced in many different contexts; for τ-sheaves [TaW], τ-crystals [BöP, Böc], Drinfeld modules [Tae1, Mor], t-modules [Fan, ANT] and shtukas [Laf]. They all compare to the above series, although we decided to remain quiet on details for the sake of concision.

Most of the known properties of these L-series are deduced from certain versions of the Woods hole trace formula, as written by Anderson in [And] (*e.g.* see Lafforgue [Laf], Taelman [Tae2], Böckle-Pink [BöP] for direct applications). In the present text, we explain how Anderson's trace formula is also well-suited to provide a fast-converging algorithm to compute the L-series of a t-motive.

A crucial ingredient for this is the notion of maximal model introduced and studied in [Gaz]. Indeed, Anderson formula cannot be applied directly on a t-motive $\underline{M}$ but requires its maximal model. In the present article, we first design an algorithm for computing the maximal model. We then design a second algorithm, mostly based on Anderson formula, to compute the L-series of $\underline{M}$ from the knowledge of its maximal model. We analyze the complexity of this second algorithm, obtaining eventually the following theorem.

Theorem (*cf* Theorem 2.2.5)**.** *There exists an algorithm taking as input*

- *the maximal model of a t-motive $\underline{M}$ of rank r (encoded by a matrix defining the action of τ_M),*
- *a place v of $\mathbb{F}[t]$ of degree d,*
- *a target precision* prec

and outputs the L-series $L_v(\underline{M}, T)$ at precision v^{prec} for a cost of

$$O\tilde{}\left(r^\Omega \cdot \left(\frac{d_\theta}{q-1} + d\right)^\Omega \cdot d \cdot \mathrm{prec}\right)$$

operations in $\mathbb{F}$ where d_θ is the maximal θ-degree of an entry of the matrix of τ_M and Ω denotes a feasible exponent of the computation of the characteristic polynomial over an abstract ring.

Along the way, we encountered the following theorem which seems to be new.

Theorem (*cf* Theorems 1.4.1 and 2.2.6)**.** *Let $\underline{M}$ be a t-motive over $\mathbb{F}(\theta)$ and let v be a place of $\mathbb{F}[t]$ (finite or infinite). Let $a_n \in \mathbb{F}(t)$ denotes the nth coefficient of $L_v(\underline{M}, T)$ as a power series in T. Then,*

$$|a_n|_v = O\left(q^{-\deg v \cdot c^n}\right), \quad \text{where} \quad c = q^{1/(\mathrm{rank}\,\underline{M})(\deg v)} > 1.$$

In particular $L_v(\underline{M}, T)$ has infinite v-adic radius of convergence. If, in addition, $\underline{M}$ is effective[1]*, then $L_v(\underline{M}, T)$ is a polynomial with coefficients in $\mathbb{F}[\theta]$.*

We hope that computations in mass using our algorithm might shed new lights on conjectural behavior of function fields special L-values. For instance, using it at different finite places v, we observed the following pattern which does not seem to be known.

Conjecture. *Assume that $\underline{M}$ has good reduction. The order of vanishing of $L_v(\underline{M}, T)$ at $T = 1$ is independent of the finite place v.*

[1]Meaning that τ_M is regular along the divisor $t = \theta$, see Definition 1.1.1 below.

We did not run the algorithm for $\underline{M}$ not having good reduction. We wish to communicate on this in a near future.

1. Anderson t-motives and their L-series

1.1. **Anderson t-motives.** Throughout this article, we fix a finite field $\mathbb{F}$ and denote by q its cardinality. We write $A = \mathbb{F}[t]$ and $K = \mathrm{Frac}\, A = \mathbb{F}(t)$. A *place ℓ of K* is, by definition, a discrete valuation subring $A_{(\ell)}$ of K whose fraction field is K. Concretely, a place of K is either the place at infinity, denoted by ∞, or a finite place given by an irreducible monic polynomial in A. To ℓ as above, one attaches:

(1) a discrete valuation v_ℓ on K, taking nonnegative values on $A_{(\ell)}$,
(2) its maximal ideal $\mathfrak{m}_\ell := \{x \in K \mid v_\ell(x) > 0\}$ (also denoted by ℓ if finite),
(3) its residue field $\mathbb{F}_\ell := A_{(\ell)}/\mathfrak{m}_\ell$ which is a finite extension of $\mathbb{F}$,
(4) its degree $\deg \ell$ which is the degree of the field extension $\mathbb{F}_\ell$ over $\mathbb{F}$,
(5) the completion A_ℓ of $A_{(\ell)}$, and K_ℓ of K.

A ring of great importance to define t-motives is the tensor product[2] $A \otimes A$. By convention, we continue to write t for $t \otimes 1$ and we set $\theta = 1 \otimes t \in A \otimes A$. With this notation, $A \otimes A$ becomes isomorphic to the ring of bivariate polynomials $\mathbb{F}[t, \theta]$. We shall also often consider the tensor product $A \otimes K$ which, using the same convention, becomes isomorphic to $\mathbb{F}(\theta)[t]$.

We let τ be the endomorphism of $A \otimes K$ acting on A by the identity and on K by the Frobenius $x \mapsto x^q$. If M is a module over $A \otimes K$, we set $\tau^\star M = (A \otimes K) \otimes_{\tau, A \otimes K} M$ where the notation means that we consider $A \otimes K$ as an algebra over itself *via* τ. For $m \in M$, we also note by $\tau^\star m$ the element $(1 \otimes m) \in \tau^\star M$.

Definition 1.1.1. An *Anderson t-motive* is the datum $\underline{M} = (M, \tau_M)$ of a finite free module M over $A \otimes K$ together with an $(A \otimes K)$-linear isomorphism

$$\tau_M : (\tau^\star M)\left[\tfrac{1}{t-\theta}\right] \xrightarrow{\sim} M\left[\tfrac{1}{t-\theta}\right].$$

The rank of M is called the *rank of* $\underline{M}$. We say that $\underline{M}$ is *effective* if τ_M factors as a map $\tau^\star M \to M$.

We fix a separably closed field K^s of K with (profinite) Galois group G_K. Let $\mathfrak{p}$ be a finite place of K. Given a place $\wp$ of K^s above $\mathfrak{p}$ (that is, a valuation ring $\mathcal{O}_{(\wp)}$ of K^s whose field of fractions is K^s and which contains $\mathcal{O}_{(\mathfrak{p})}$ as a sub-valution ring), we introduce the following subgroups of G_K:

(1) The decomposition subgroup $D_\wp := \{\sigma \in G_K \mid \sigma(\mathcal{O}_{(\wp)}) = \mathcal{O}_{(\wp)}\}$,
(2) The inertia subgroup $I_\wp := \{\sigma \in G_K \mid \sigma(\wp) = \wp\} \subset D_\wp$, where $\wp$ is the maximal ideal of $\mathcal{O}_{(\wp)}$.

The group $D_\wp$ also identifies with the profinite Galois group $\mathrm{Gal}(K^s_\wp | K_\mathfrak{p})$ where $K^s_\wp$ denotes the completion of F^s with respect to the valuation $v_{(\wp)}$. We also recall that the inclusion $I_\wp \subset D_\wp$ only depends on $\mathfrak{p}$ up to conjugation, and that it sits in a short exact sequence:

$$0 \longrightarrow I_\wp \longrightarrow D_\wp \longrightarrow G_{\mathbb{F}_\mathfrak{p}} \longrightarrow 0$$

where $G_{\mathbb{F}_\mathfrak{p}}$ is the absolute Galois group of the residue field $\mathbb{F}_\mathfrak{p}$. We denote by $\mathrm{Frob}_\mathfrak{p}$ the Frobenius endomorphism, *i.e.* $x \mapsto x^{q^d}$ where $d = \deg \mathfrak{p}$.

[2] Throughout this article, unlabeled tensor product will implicitly be taken over $\mathbb{F}$.

Let $\underline{M}$ be an t-motive of rank $r \geq 0$. Let ℓ be a finite place of A; we attach to $\underline{M}$ a continuous ℓ-adic representation of $D_\wp$ (see [Gaz, Definition 2.24]):

$$\mathrm{T}_\ell\,\underline{M} := \varprojlim_n (M/\ell^n M \otimes_K K_\wp^s)^{\tau_M=1}.$$

Proposition 2.26 in *loc. cit.* implies that $\mathrm{T}_\ell\,\underline{M}$ defines a rank r continuous representation of $D_\wp$. As a consequence, the group $G_{\mathbb{F}_\mathfrak{p}}$ acts on $(\mathrm{T}_\ell\,\underline{M})^{I_\wp}$ continuously. This allows to introduce the local L-factor at $\mathfrak{p}$ relative to ℓ.

Definition 1.1.2. The *local L-factor of $\underline{M}$ at $\mathfrak{p}$ and relative to ℓ* is the polynomial

$$P_\mathfrak{p}(\mathrm{T}_\ell\,\underline{M}, T) := \det\nolimits_{A_\ell}\left(\mathrm{id} - T^{\deg \mathfrak{p}}\mathrm{Frob}_\mathfrak{p}^{-1} | (\mathrm{T}_\ell\,\underline{M})^{I_\wp}\right) \in A_\ell[T].$$

Remark 1.1.3. As the notation suggests, the polynomial $P_\mathfrak{p}(\mathrm{T}_\ell\,\underline{M}, T)$ depends on $\mathfrak{p}$ (but not on $\wp$). This is because the determinant is invariant under conjugation. We will show next that it is independent of ℓ.

1.2. **A formula for the local L-factor.** In this subsection, we give a formula for the local L-factor in terms of the maximal integral model. This will show on the one hand that the local L-factor of $\underline{M}$ at $\mathfrak{p}$ is independent of ℓ (as long as $\mathfrak{p}$ is not above ℓ) and has coefficients in K and, on the other hand, this will allow us to apply Anderson's trace formula.

1.2.1. *Maximal models.* Let $\underline{M}$ be a t-motive over K. We recall the notion of lattices and models of $\underline{M}$ following [Gaz, §4].

Definition 1.2.1. Let N be a sub-$A \otimes A$-module of M.

(1) N is called a *lattice* if it is finitely generated and generates M over K.
(2) N is called *stable by τ_M* (or *stable* for short) if $\tau_M(\tau^\star N) \subset N[(t-\theta)^{-1}]$.
(3) If N is both a lattice and stable, then we say that N is a *model* of $\underline{M}$.
(4) If N is a model of $\underline{M}$, we call N *maximal* if it is not strictly contained in any other model.

Note that a model of $\underline{M}$ need not to be free; if it does, we call it a *free model*.

Models of t-motives are in some sense analogue to integral models of varieties defined over number fields. The following result, reminiscent of the existence of Néron models for abelian varieties, was proven in *loc. cit.* (combination of Proposition 4.30, Theorem 4.32 there).

Theorem 1.2.2. *A maximal model for $\underline{M}$ exists and is unique. In addition, it is projective over $A \otimes A$.*

Remark 1.2.3. Since $A \otimes A \simeq \mathbb{F}[t, \theta]$ is a bivariate polynomial ring over a field, Quillen–Suslin's theorem [Qui] shows that the maximal model of $\underline{M}$ is necessarily free. We shall give in Section 2.1 a direct proof of this freeness property (including an algorithm for computing a basis) avoiding the use of the profound and complicated theorem of Quillen and Suslin.

To a free model N, we associate its *discriminant* $\Delta_N \in \mathbb{F}[\theta]$ defined as follow. If a basis $\mathbf{b}$ of N over $\mathbb{F}[t, \theta]$ is given, the determinant of the matrix representing τ_M in $\tau^\star\mathbf{b}$ and $\mathbf{b}$ depends on the choice of $\mathbf{b}$ only up to a unit. The ideal of $\mathbb{F}[t,\theta][(t-\theta)^{-1}]$ it generates is of the form $(t-\theta)^h \cdot \mathfrak{d}_N$ for an integer h depending solely on $\underline{M}$ and a non zero ideal $\mathfrak{d}_N \subset \mathbb{F}[\theta]$ depending solely on N. The monic generator of $\mathfrak{d}_N$ is called the *discriminant of* N.

Definition 1.2.4. We call the *discriminant of* $\underline{M}$ and denote by $\Delta_{\underline{M}}$ the discriminant of $M_\mathcal{O}$. We say that $\underline{M}$ has *good reduction* whenever $\Delta_{\underline{M}} = 1$.

Relying on Theorem 1.2.2, one may introduce another local L-factor. Given two distinct finite places $\mathfrak{p}$ and ℓ of A, we define

$$P_{\mathfrak{p}}(M_{\mathcal{O}}, T) := \det_{A[\mathfrak{p}^{-1}]}\left(\mathrm{id} - T\tau_M | M_{\mathcal{O}} \otimes_{A\otimes A} (A[\mathfrak{p}^{-1}] \otimes \mathbb{F}_{\mathfrak{p}})\right) \in A[\mathfrak{p}^{-1}][T].$$

The determinant is well-defined as it is taken over a free $A[\mathfrak{p}^{-1}]$-module of rank $(\operatorname{rank} \underline{M})(\deg \mathfrak{p})$.

Theorem 1.2.5. *For any two distinct finite places $\mathfrak{p}$ and ℓ of A, we have $P_{\mathfrak{p}}(\mathrm{T}_\ell \underline{M}, T) = P_{\mathfrak{p}}(M_{\mathcal{O}}, T)$.*

This has the following immediate corollary which implies the well-definedness of the infinite product we wrote earlier in (1).

Corollary 1.2.6. *The polynomial $P_{\mathfrak{p}}(\mathrm{T}_\ell \underline{M}, T)$ is independent of ℓ and belongs to $1 + T^d K[T^d]$ where $d = \deg \mathfrak{p}$.*

1.2.2. *Frobenius spaces.* Before elaborating on the proof of Theorem 1.2.5, we need to review some facts about Frobenius spaces and their models. They are much easier to handle than t-motives and their model, and we will prove Theorem 1.2.5 by reducing to the case of Frobenius spaces. We follow [Gaz, §4.1] closely.

We consider a finite place $\mathfrak{p}$ of K. As before, we let $\mathcal{O}_{\mathfrak{p}}$ (resp. $K_{\mathfrak{p}}$) be the completion of A (resp. of K) at $\mathfrak{p}$ and we fix a separable closure $K_{\mathfrak{p}}^s$ of it. We denote by $G_{\mathfrak{p}}$ the absolute Galois group $\mathrm{Gal}(K_{\mathfrak{p}}^s | K_{\mathfrak{p}})$ equipped with its profinite topology, and we let $I_{\mathfrak{p}} \subset G_{\mathfrak{p}}$ be the inertia subgroup. Let $\sigma : K_{\mathfrak{p}} \to K_{\mathfrak{p}}$ be the q-Frobenius.

A *Frobenius space* over $K_{\mathfrak{p}}$ is a pair (V, φ) where V is a finite dimensional $K_{\mathfrak{p}}$-vector space and $\varphi : \sigma^\star V \to V$ is an $K_{\mathfrak{p}}$-linear isomorphism. By an *$\mathcal{O}_{\mathfrak{p}}$-lattice in V*, we mean a finitely generated $\mathcal{O}_{\mathfrak{p}}$-submodule N of N which generates V over $K_{\mathfrak{p}}$. We say that a $\mathcal{O}_{\mathfrak{p}}$-submodule L *is stable by* φ if $\varphi(\sigma^\star N) \subset N$.

Definition 1.2.7. Let N be a finitely generated $\mathcal{O}_{\mathfrak{p}}$-submodule of V.

(1) We say that N is an *integral model for* (V, φ) if N is an $\mathcal{O}_{\mathfrak{p}}$-lattice in V stable by φ. We say that N is *maximal* if it is not strictly included in another integral model for (V, φ).
(2) We say that N is a *good model for* (V, φ) if $\varphi(\sigma^\star L) = L$. We say that N is *maximal* if it is not strictly included in another good model of (V, φ).

The following was proven in Proposition 4.2 and Lemma 4.11 of [Gaz].

Proposition 1.2.8. *A maximal model (resp. maximal good model) for (V, φ) exists and is unique.*

Definition 1.2.9. We denote by $V_{\mathcal{O}}$ the maximal integral model of (V, φ). We denote by V_{good} the maximal good model of (V, φ). We say that (V, φ) has *good reduction* if the inclusion $V_{\text{good}} \subseteq V_{\mathcal{O}}$ is an equality.

The relation among $V_{\mathcal{O}}$ and V_{good} can be described as follow.

Proposition 1.2.10. *The module $V_{\mathcal{O}}$ decomposes as $V_{\text{good}} \oplus V_{\text{nil}}$, where $V_{\text{nil}} \subset V$ is a sub-$\mathcal{O}$-module stable by φ and on which φ is topologically nilpotent.*

Frobenius spaces are directly related to Galois representations thanks to Katz's equivalence of categories, recalled below.

Theorem 1.2.11 ([Kat2], Proposition 4.1.1)**.** *There is a rank-preserving equivalence of categories from the category of Frobenius spaces over $K_{\mathfrak{p}}$ and the category of $\mathbb{F}$-linear continuous representation of $G_{\mathfrak{p}}$, explicitly given by the functor*

$$\mathrm{T} : \underline{V} = (V, \varphi) \longmapsto (V \otimes_{K_{\mathfrak{p}}} K_{\mathfrak{p}}^s)^{\varphi=1} := \left\{x \in V \otimes_{K_{\mathfrak{p}}} K_{\mathfrak{p}}^s \mid x = \varphi(\sigma^\star x)\right\}$$

where $G_{\mathfrak{p}}$ *acts on* $\mathrm{T}\,\underline{V}$ via *its action on* $K_{\mathfrak{p}}^s$.

Let B be a finite commutative $\mathbb{F}$-algebra. Noticing that a B-linear representation is nothing but a $\mathbb{F}$-linear representation endowed with an additional action of B, we deduce from Katz's theorem that the functor T induces another equivalence of categories:

$$\left\{\begin{matrix}\text{Frobenius spaces over } K_{\mathfrak{p}} \\ \text{with coefficients in } B\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{continous } B\text{-linear} \\ \text{representations of } G_{\mathfrak{p}}\end{matrix}\right\}$$

where, by a Frobenius space over $K_{\mathfrak{p}}$ with coefficients in B, we mean a finite $B\otimes K_{\mathfrak{p}}$-module V equipped with an $B\otimes K_{\mathfrak{p}}$-linear isomorphism $\varphi:\tau^\star V\xrightarrow{\sim} V$ where we denoted $\tau=\mathrm{id}\otimes\sigma$ for consistency.

1.2.3. *Local L-factors of Frobenius spaces.* Katz's equivalence relates two objects to which one can associate a local L-factor. Our aim is to show that the L-factors are preserved under the equivalence. Any Frobenius space (V,φ) over $K_{\mathfrak{p}}$ with coefficients in B induces a Frobenius space over $K_{\mathfrak{p}}$ by forgetting the action of B. In that respect, it admits a maximal integral model $V_{\mathcal{O}}$ as well as a maximal good model V_{good}. By functoriality of the assignment $V\mapsto V_{\mathcal{O}}$ (resp. $V\mapsto V_{\mathrm{good}}$), both $V_{\mathcal{O}}$ and V_{good} are canonically $B\otimes K_{\mathfrak{p}}$-modules.

Definition 1.2.12. (1) Let $\underline{V}=(V,\varphi)$ be a Frobenius space over $K_{\mathfrak{p}}$ with coefficients in B with maximal integral model $V_{\mathcal{O}}$. The polynomial

$$P(\underline{V},T)=\det{}_B\left(\mathrm{id}-T\varphi|V_{\mathcal{O}}/\mathfrak{p}V_{\mathcal{O}}\right)\in 1+TB[T]$$

is called the *local L-factor of* $\underline{V}$.

(2) Let $\underline{\mathrm{T}}$ be a B-linear continuous representation of $G_{\mathfrak{p}}$. Let d be the degree of $\mathfrak{p}$. The polynomial

$$P(\underline{\mathrm{T}},T)=\det{}_B\left(\mathrm{id}-T^d\mathrm{Frob}_{\mathfrak{p}}^{-1}|\underline{\mathrm{T}}^{I_{\mathfrak{p}}}\right)\in 1+T^dB[T^d]$$

is called the *local L-factor of* $\underline{\mathrm{T}}$.

Let $\underline{V}$ be a Frobenius space, and let $\mathrm{T}\,\underline{V}$ be the B-linear continuous representation of $G_{\mathfrak{p}}$ associated to it through Katz's equivalence. A key step to prove Theorem 1.2.5 is the following proposition.

Proposition 1.2.13. $P(\underline{V},T)=P(\mathrm{T}\,\underline{V},T)$.

Proof. First observe that, by Katz's equivalence over $\mathcal{O}_{\mathfrak{p}}$ (as in the proof of [Gaz, Proposition 4.17]), the representation $\mathrm{T}\,\underline{V}^{I_{\mathfrak{p}}}$ corresponds to $(V_{\mathrm{good}}\otimes_{\mathcal{O}_{\mathfrak{p}}}\bar{\mathcal{O}}_{\mathfrak{p}})^{\varphi=1}$.
Fixing a basis of L as an $\mathcal{O}_{\mathfrak{p}}$-module, and writing the action of φ in this basis as a matrix $G\in\mathrm{Mat}_r(\bar{\mathcal{O}}_{\mathfrak{p}})$, we recognize that the reduction map $(V_{\mathrm{good}}\otimes_{\mathcal{O}_{\mathfrak{p}}}\bar{\mathcal{O}}_{\mathfrak{p}})^{\varphi=1}\to(V_{\mathrm{good}}\otimes_{\mathcal{O}_{\mathfrak{p}}}\bar{\mathbb{F}}_{\mathfrak{p}})^{\varphi=1}$ reduces to a map $\mathrm{red}:X(\bar{\mathcal{O}}_{\mathfrak{p}})\to X(\bar{\mathbb{F}}_{\mathfrak{p}})$ where

$$X(S):=\left\{(x_1,\dots,x_r)\in S^r|(x_1,\dots,x_r)-(x_1^q,\dots,x_r^q)G=0\right\}.$$

By the multivariate Hensel's lemma, red is a bijection. Therefore

$$\mathrm{T}\,\underline{V}^{I_{\mathfrak{p}}}=(V_{\mathrm{good}}\otimes_{\mathcal{O}_{\mathfrak{p}}}\bar{\mathbb{F}}_{\mathfrak{p}})^{\varphi=1}, \tag{2}$$

where $\mathrm{Frob}_{\mathfrak{p}}^{-1}$ on the left-hand side acts as $(1\otimes\sigma^{-d})=(\varphi^d\otimes 1)$ on the right-hand side.
On the other hand, due to Proposition 1.2.10, we have

$$V_{\mathcal{O}}/\mathfrak{p}V_{\mathcal{O}}\cong(V_{\mathrm{good}}/\mathfrak{p}V_{\mathrm{good}})\oplus(V_{\mathrm{nil}}/\mathfrak{p}V_{\mathrm{nil}}),$$

and the map φ is nilpotent once restricted to the right-hand summand. This decomposition further respects the structure of B-modules. Hence,

$$\det{}_B\left(\mathrm{id}-T\varphi|V_{\mathcal{O}}/\mathfrak{p}V_{\mathcal{O}}\right)=\det{}_B\left(\mathrm{id}-T\varphi|V_{\mathrm{good}}/\mathfrak{p}V_{\mathrm{good}}\right).$$

Therefore,

$$
\begin{aligned}
P(\underline{V},T) &= \det_B\left(\mathrm{id}-T(\varphi\otimes_{\mathcal{O}_\mathfrak{p}}\sigma)|V_{\text{good}}\otimes_{\mathcal{O}_\mathfrak{p}}\mathbb{F}_p\right)\\
&= \det_{B\otimes\mathbb{F}_\mathfrak{p}}\left(\mathrm{id}-T^d\varphi^d|V_{\text{good}}\otimes_{\mathcal{O}_\mathfrak{p}}\mathbb{F}_\mathfrak{p}\right) &(3)\\
&= \det_{B\otimes\bar{\mathbb{F}}_\mathfrak{p}}\left(\mathrm{id}-T^d\varphi^d\otimes_{\mathbb{F}_\mathfrak{p}}\mathrm{id}|V_{\text{good}}\otimes_{\mathcal{O}_\mathfrak{p}}\mathbb{F}_\mathfrak{p}\otimes_{\mathbb{F}_\mathfrak{p}}\bar{\mathbb{F}}_\mathfrak{p}\right) &(4)\\
&= \det_{B\otimes\bar{\mathbb{F}}_\mathfrak{p}}\left(\mathrm{id}-T^d(\varphi^d\otimes_{\mathcal{O}_\mathfrak{p}}\mathrm{id})\otimes\mathrm{id}|(V_{\text{good}}\otimes_{\mathcal{O}_\mathfrak{p}}\bar{\mathbb{F}}_\mathfrak{p})^{\varphi=1}\otimes\bar{\mathbb{F}}_\mathfrak{p}\right) &(5)\\
&= \det_B\left(\mathrm{id}-T^d(\varphi^d\otimes_{\mathcal{O}_\mathfrak{p}}\mathrm{id})|(V_{\text{good}}\otimes_{\mathcal{O}_\mathfrak{p}}\bar{\mathbb{F}}_\mathfrak{p})^{\varphi=1}\right) &(6)\\
&= P(\mathrm{T}\,\underline{V},T) &(7)
\end{aligned}
$$

where in (3) we used *e.g.* [BöP, Lemma 8.1.4], in (4) we extended linearly scalars through $\mathbb{F}_\mathfrak{p}\to\bar{\mathbb{F}}_\mathfrak{p}$, in (5) we used Lang isogeny theorem [Kat1, Proposition 1.1], in (6) we descended through $\mathbb{F}\to\bar{\mathbb{F}}_\mathfrak{p}$, and in (7) we used the equality (2). ☐

1.2.4. *Proof of Theorem 1.2.5.* Let n be a positive integer and let ℓ be a finite place of K distinct form $\mathfrak{p}$. Let $M_\mathfrak{p}$ denote the base-change $M\otimes_K K_\mathfrak{p}$. The datum of $\underline{V}=(M_\mathfrak{p}/\ell^n M_\mathfrak{p},\tau_M)$ defines a Frobenius space over $K_\mathfrak{p}$ with coefficients in A/ℓ^n. As such, it admits a maximal model $V_\mathcal{O}$ which is a module over $A/\ell^n\otimes A$, and Proposition 1.2.13 gives the equality

$$P(\mathrm{T}\,\underline{V},T)=P(\underline{V},T)$$

as polynomials in $A/\ell^n[T]$. Note that the left-hand side identifies with the reduction modulo ℓ^n of $\det_{A_\ell}(\mathrm{id}-T^{\deg\mathfrak{p}}\mathrm{Frob}_\mathfrak{p}|(\mathrm{T}_\ell\,\underline{M})^{I_\mathfrak{p}})$, while the right-hand side corresponds to $\det_{A/\ell^n}(\mathrm{id}-T\tau_M|V_\mathcal{O}\otimes_A\mathbb{F}_\mathfrak{p})$.

We set $M_{\mathcal{O}_\mathfrak{p}}:=M_\mathcal{O}\otimes_{A\otimes A}(A\otimes\mathcal{O}_\mathfrak{p})$. Note that we cannot conclude yet, as $V_\mathcal{O}$ might not correspond to $M_{\mathcal{O}_\mathfrak{p}}/\ell^n M_{\mathcal{O}_\mathfrak{p}}$. However, we have

$$M_{\mathcal{O}_\mathfrak{p}}+\ell^{k_n}M_\mathfrak{p}=V_\mathcal{O}+\ell^{k_n}M_\mathfrak{p}$$

for some integer $k_n\leq n$ which tends to infinity as n does; this results first from [Gaz, Theorem 4.55] to identify $M_{\mathcal{O}_\mathfrak{p}}$ with the maximal model of $\underline{M}_\mathfrak{p}$, then from [Gaz, Lemma 4.47]. Therefore, we obtain

$$\det_{A_\ell}(\mathrm{id}-T^{\deg\mathfrak{p}}\mathrm{Frob}_\mathfrak{p}^{-1}|(\mathrm{T}_\ell\,\underline{M})^{I_\mathfrak{p}})\equiv\det_{A_\ell}(\mathrm{id}-T\tau_M|(M_\mathcal{O}\otimes_{A\otimes A}(A\otimes\mathbb{F}_\mathfrak{p}))^\wedge_\ell)\pmod{\ell^{k_n}}.$$

But since the endomorphism τ_M acting on $(M_\mathcal{O}\otimes_{A\otimes A}(A\otimes\mathbb{F}_\mathfrak{p}))^\wedge_\ell$ is induced by its $A[\mathfrak{p}^{-1}]$-linear action on $M_\mathcal{O}\otimes_{A\otimes A}(A[\mathfrak{p}^{-1}]\otimes\mathbb{F}_\mathfrak{p})$, we get

$$\det_{A_\ell}(\mathrm{id}-T^{\deg\mathfrak{p}}\mathrm{Frob}_\mathfrak{p}^{-1}|(\mathrm{T}_\ell\,\underline{M})^{I_\mathfrak{p}})\equiv\det_{A[\mathfrak{p}^{-1}]}(\mathrm{id}-T\tau_M|M_\mathcal{O}\otimes_{A\otimes A}(A[\mathfrak{p}^{-1}]\otimes\mathbb{F}_\mathfrak{p}))\pmod{\ell^{k_n}}.$$

Taking the limit as n tends to infinity achieves the proof of Theorem 1.2.5.

1.3. Anderson trace formula.

We first recall the notion of *nuclear operator* introduced by Anderson in [And] and its extension with general coefficient ring by Böckle–Pink in [BöP]. Let B be an $\mathbb{F}$-algebra, let D be a free B-module and let $\kappa:D\to D$ be a B-linear map. By freeness, one can write $D=B\otimes D_0$ (as B-modules) for some $\mathbb{F}$-vector space D_0. Upon the choice of D_0, Böckle and Pink introduce the following definition.

Definition 1.3.1. An $\mathbb{F}$-vector space $W_0\subset D_0$ is called *a nucleus for* κ if it finite dimensional and there exists an exhaustive increasing filtration of N_0 by finite dimensional $\mathbb{F}$-vector spaces $W_0\subset W_1\subset W_2\subset\cdots$ such that $\kappa(B\otimes W_{i+1})\subset B\otimes W_i$ for all $i>0$.
If a nucleus for κ exists, we call κ *nuclear.*

Proposition 1.3.2 ([BöP], Propositions 8.3.2 & 8.3.3)**.** *Suppose that κ admits a nucleus W_0. Then, the expressions*

$$\mathrm{Tr}(\kappa):=\mathrm{Tr}_B(\kappa|B\otimes W_0)\in B,$$
$$\Delta(\mathrm{id}-T\kappa):=\det_B(\mathrm{id}-T\kappa|B\otimes W_0)\in B[T],$$

are independent of the nucleus W_0.

We will call $\mathrm{Tr}(\kappa)$ and $\Delta(\mathrm{id} - T\kappa)$ the *trace* and the *dual characteristic polynomial of* κ.

Remark 1.3.3. Note that although the trace (resp. dual characteristic polynomial) of κ does not depend on the nucleus W_0, this might still depends on the decomposition $D = B \otimes D_0$.

We express Anderson trace formula in the form recorded by Böckle–Pink in [BöP, §8.3]. Let B and R be $\mathbb{F}$-algebras and assume that R is a Dedekind domain. We denote by $\sigma : R \to R$ the q-Frobenius and $\tau = \mathrm{id} \otimes \sigma$ the B-linear ring endomorphism of $B \otimes R$ which acts as the q-Frobenius on R.

Let $\Omega^1_R = \Omega^1_{R/\mathbb{F}}$ be the R-module of Kähler differentials of R relative to $\mathbb{F}$. The *q-Cartier operator* on Ω^1_R is a R-linear map $C : \Omega^1_R \to \sigma^\star \Omega^1_R$ uniquely determined to have the following properties for any $a \in R$:

(1) $C(da) = 0$,
(2) $C(a^{q-1}da) = da$

Let N be a finite free $B \otimes R$-module and let $\tau_N : \tau^\star N \to N$ be a linear morphism. To the datum of (N, τ_N), one associates another $(N^\star, \tau_N^\star)$. The $B \otimes R$-module $N^\star$ is the *Kähler dual* of N:

$$N^\star = \mathrm{Hom}_{B\otimes R}(N, B \otimes \Omega^1_R).$$

The morphism $\tau_N^\star : N^\star \to \tau^\star N^\star$ is given on $f \in N$ by the expression:

$$\tau_N^\star(f) := C \circ f \circ \tau_N.$$

We fix an identification $N = B \otimes N_0$ for some finite R-module N_0. This induces an identification $N^\star = B \otimes N_0^\star$ where $N_0^\star = \mathrm{Hom}_R(N_0, \Omega^1_R)$. It follows from [BöP, Proposition 8.3.8] that $\tau_N^\star$ is nuclear and admits a nucleus $W_0 \subset N_0^\star$.

Theorem 1.3.4 ([BöP], Theorem 8.3.10, Anderson trace formula)**.** *In the ring $B[[T]]$, we have*

$$\prod_{\mathfrak{p}} \det{}_B(\mathrm{id} - T\tau_N | N \otimes_R R/\mathfrak{p})^{-1} = \Delta(\mathrm{id} - T\tau_N^\star)$$

where the product on the left run over all maximal ideals $\mathfrak{p}$ in R and the dual characteristic polynomial on the right is taken with respect to the decomposition $N^\star = B \otimes N_0^\star$.

Remark 1.3.5. Since $\det_B(\mathrm{id} - T\tau_N | N \otimes_R R/\mathfrak{p}) = \det_{B\otimes\mathbb{F}_\mathfrak{p}}(\mathrm{id} - T^d\tau_N | N \otimes_R R/\mathfrak{p})$, where d is the degree of $\mathfrak{p}$, and since there are finitely many maximal ideals in R with a given degree, the formal product does make sense in $B[[T]]$.

1.4. **Global L-series.** Let $\underline{M}$ be a t-motive over K. For a place v of K (finite or infinite), we define

$$L_v(\underline{M}, T) := \prod_{\mathfrak{p} \neq v} P_\mathfrak{p}(M_\mathcal{O}, T)^{-1},$$

where the formal product is taken over finite places distinct from v (the condition is then empty if v is infinite). From Corollary 1.2.6, the product makes sense in $K[[T]]$ but also in $\mathcal{O}_v[[T]]$ when v is finite. Moreover, it follows from Theorem 1.2.5 that $L_v(\underline{M}, T)$ defined as above agree with the L-series defined in the introduction (see Equation (1)).

The decisive advantage of the second definition is that it can be approached using Anderson trace formula with the pair $(M_\mathcal{O}, \tau_M)$. This is not actually entirely correct because τ_M does not induce an $\mathbb{F}[t]$-linear endomorphism of $M_\mathcal{O}$ in full generality. This however works well when $\underline{M}$ effective in which case we get the following theorem due to Böckle [Böc] in the context of τ-crystals.

Theorem 1.4.1 (Böckle). *Let $\underline{M}$ be an effective t-motive over K. Then $L_v(\underline{M}, T)$ is a polynomial with coefficients in A.*

Proof. In the situation where $\underline{M}$ is effective, we have

$$P_{\mathfrak{p}}(M_{\mathscr{O}}, T) = \det_{A[\mathfrak{p}^{-1}]}\left(\mathrm{id} - T\tau_M | M_{\mathscr{O}} \otimes_{A\otimes A} (A[\mathfrak{p}^{-1}] \otimes \mathbb{F}_{\mathfrak{p}})\right) = \det_A (\mathrm{id} - T\tau_M | M_{\mathscr{O}} \otimes_{A\otimes A} (A \otimes \mathbb{F}_{\mathfrak{p}}))$$

as τ_M factors as an endomorphism of $M_{\mathscr{O}} \otimes_{A\otimes A} (A \otimes \mathbb{F}_{\mathfrak{p}})$. Therefore, we can apply Anderson trace formula to the pair (M, τ_M), with $B = A$ and $R = A$ if v is infinite or $R = A[v^{-1}]$ if v is finite, which implies that $L_v(\underline{M}, T)$ is a polynomial with coefficients in A. □

The effective case is unfortunately not the most interesting one. In the next sections, we will adapt the above strategy to the general case by working modulo an arbitrary positive power of v.

2. Algorithm for computing the L-series

Let $\underline{M}$ be a t-motive over $K = \mathbb{F}(\theta)$ and let v be a place of K (finite or infinite). There are two main steps to compute the L-series of $\underline{M}$. First, one should explicit its maximal model $M_{\mathscr{O}}$ and second one should find an appropriate nucleus of τ_M modulo v^n. In this section, we give algorithms for both steps.

2.1. **Determination of the maximal model.** According to Theorem 1.2.2, $\underline{M}$ admits a maximal model $M_{\mathscr{O}}$ which, in addition, is finite projective over $\mathbb{F}[t,\theta]$. By the Quillen–Suslin theorem, we know that $M_{\mathscr{O}}$ is free over $\mathbb{F}[t,\theta]$. In this subsection, we provide an algorithm computing a basis of $M_{\mathscr{O}}$.

2.1.1. *Construction of the maximal model.* We begin by giving an explicit iterative construction of $M_{\mathscr{O}}$. It proceeds in several steps.

Step 1. The first step consists in finding a model of $\underline{M}$. This is quite classical: it suffices to take any basis of M over $\mathbb{F}(\theta)[t]$ and let N' be the $\mathbb{F}[t,\theta]$-module it generates. It may not be a model, but since it is a lattice, there exists $f \in \mathbb{F}[\theta] \smallsetminus \{0\}$ for which $\tau_M(\tau^\star N') \subset \frac{1}{f} N'[(t-\theta)^{-1}]$. Then, the $\mathbb{F}[t,\theta]$-module $N := fN'$ is a model of $\underline{M}$ (*e.g.* [Gaz, Proposition 4.20]). Moreover, it is free over $\mathbb{F}[t,\theta]$.

Let $\Delta(\theta) \in \mathbb{F}[\theta]$ be the discriminant of N. We factor it into monic irreducible polynomials as

$$\Delta(\theta) = \mathfrak{p}_1(\theta)^{n_1} \cdots \mathfrak{p}_s(\theta)^{n_s}.$$

Then

$$N\left[\mathfrak{p}_1^{-1}, \ldots, \mathfrak{p}_s^{-1}\right] = M_{\mathscr{O}}\left[\mathfrak{p}_1^{-1}, \ldots, \mathfrak{p}_s^{-1}\right].$$

The strategy of the rest of the proof is to approach $M_{\mathscr{O}}$ from N by removing the $\mathfrak{p}_i$'s one by one.

Step 2. Let $\mathfrak{p} := \mathfrak{p}_s$. Let $N_0 := N$. By induction, for $i \geq 0$, we define $\mathbb{F}[t,\theta]$-modules

$$N_{i+1} := \left\{ x \in \mathfrak{p}^{-1} N_i \;\middle|\; \tau_M(\tau^\star x) \in N_i\left[\frac{1}{t-\theta}\right] \right\} \subset M.$$

We gather some properties of this sequence in the next lemma.

Lemma 2.1.1. *The following holds.*

1. *The sequence $(N_i)_{i\geq 0}$ is increasing for the inclusion.*
2. *For all $i \geq 0$, N_i is a model of $\underline{M}$.*
3. *The sequence $(N_i)_{i\geq 0}$ is stationary.*

(4) *Its limit, N_∞, is a model which satisfies*

$$N_\infty \subset M_{\mathscr{O}} \subset \mathfrak{p}^{-1} N_\infty\big[\mathfrak{p}_1^{-1}, \ldots, \mathfrak{p}_{s-1}^{-1}\big].$$

Proof. We prove (1) by induction on i. It is clear that $N_0 \subset N_1$. Assuming $N_{i-1} \subset N_i$ for some i, and taking $x \in N_i$, we get $\tau_M(\tau^\star x) \in N_{i-1}[(t-\theta)^{-1}] \subset N_i[(t-\theta)^{-1}]$ and hence $x \in N_{i+1}$. That is $N_i \subset N_{i+1}$. We deduce both from (1) that N_i is stable and that it is generating (as it contains N_0). This implies (2).
By maximality we then get $N_i \subset M_{\mathscr{O}}$ and hence, by noetherianity, the sequence $(N_i)_{i\geq 0}$ stabilizes, proving (3).
It remains to show (4). Because it equals N_I for I large, N_∞ is a model. Let n be the minimal integer such that $M_{\mathscr{O}} \subset \mathfrak{p}^{-n} N_\infty\big[\mathfrak{p}_1^{-1}, \ldots, \mathfrak{p}_{s-1}^{-1}\big]$ and suppose that $n \geq 2$. There exists $m \in M_{\mathscr{O}}$ such that $\mathfrak{p}^n m \in N_\infty\big[\mathfrak{p}_1^{-1}, \ldots, \mathfrak{p}_{s-1}^{-1}\big]$ but $\mathfrak{p}^{n-1} m$ does not. Its image under τ_M is so that

$$\tau_M(\tau^\star \mathfrak{p}^{n-1} m) = \mathfrak{p}^{q(n-1)} \tau_M(\tau^\star m) \in N_\infty\left[\frac{1}{t-\theta}, \mathfrak{p}_1^{-1}, \ldots, \mathfrak{p}_{s-1}^{-1}\right].$$

In particular, there exists $e \in \mathbb{F}[\theta]$ supported at $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_{s-1}\}$, such that $\tau_M(\tau^\star \mathfrak{p}^{n-1} e m) \in N_\infty$. By design, this implies $\mathfrak{p}^{n-1} e m \in N_\infty$ and thus $\mathfrak{p}^{n-1} m \in N_\infty\big[\mathfrak{p}_1^{-1}, \ldots, \mathfrak{p}_{s-1}^{-1}\big]$. This is in contradiction with the choice of m. □

Step 3. Let $L_0 = \mathfrak{p}^{-1} N_\infty$. Starting from L_0, we define another sequence of $\mathbb{F}[t,\theta]$-modules $(L_i)_{i\geq 0}$ by the expression

$$L_{i+1} \coloneqq \left\{ x \in L_i \;\middle|\; \tau_M(\tau^\star x) \in L_i\left[\frac{1}{t-\theta}\right] \right\} \subset M.$$

We record properties of $(L_i)_{i\geq 0}$ in a lemma.

Lemma 2.1.2. *For all $i \geq 0$,*

(1) $N_\infty \subset L_{i+1} \subset L_i$,
(2) $\mathfrak{p} L_i \subset L_{i+1}$,
(3) $\mathfrak{p} L_i$ *is a model; in particular* $\mathfrak{p} L_i \subset M_{\mathscr{O}}$,
(4) $M_{\mathscr{O}} \subset L_i\big[\mathfrak{p}_1^{-1}, \ldots, \mathfrak{p}_{s-1}^{-1}\big]$,
(5) *the sequence* $(L_i)_{i\geq 0}$ *is stationary,*
(6) *if* L_∞ *denotes the limit, then* L_∞ *is a model of* $\underline{M}$ *and*

$$L_\infty\big[\mathfrak{p}_1^{-1}, \ldots, \mathfrak{p}_{s-1}^{-1}\big] = M_{\mathscr{O}}\big[\mathfrak{p}_1^{-1}, \ldots, \mathfrak{p}_{s-1}^{-1}\big].$$

Proof. Point (1) follows from an immediate induction, using that N_∞ is a model of $\underline{M}$. Point (2) for $i = 0$ is clear. Assuming $\mathfrak{p} L_{i-1} \subset L_i$ for $i > 0$ and taking $x \in \mathfrak{p} L_i$, we can write $x = \mathfrak{p} \cdot l_i$ for $l_i \in L_i$ so that

$$\tau_M(\tau^\star x) = \mathfrak{p}^q \tau_M(\tau^\star l_i) \in \mathfrak{p}^q L_{i-1}\left[\frac{1}{t-\theta}\right] \subset L_i\left[\frac{1}{t-\theta}\right],$$

and hence $x \in L_{i+1}$ as desired. Given that $\mathfrak{p}^q L_{i-1} \subset \mathfrak{p} L_i$, the previous computation shows also that $\mathfrak{p} L_i$ is a model, hence point (3).
Point (4) follows from induction as well: initialization is Lemma 2.1.1.(4). Assuming that the result holds at $i > 0$, and given $x \in M_{\mathscr{O}}$,

$$\tau_M(\tau^\star x) \in M_{\mathscr{O}}\left[\frac{1}{t-\theta}\right] \subset L_i\left[\frac{1}{t-\theta}, \mathfrak{p}_1^{-1}, \ldots, \mathfrak{p}_{s-1}^{-1}\right],$$

and hence $x \in L_{i+1}\big[\mathfrak{p}_1^{-1}, \ldots, \mathfrak{p}_{s-1}^{-1}\big]$ (more precisely, there exists $e \in \mathbb{F}[\theta] \setminus \{0\}$ supported at $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_{s-1}\}$ such that $\tau_M(\tau^\star e x) \in L_i[(t-\theta)^{-1}]$, hence $ex \in L_{i+1}$).
To show (5), it suffices to notice that the sequence $(L_i)_{i\geq 0}$ is a decreasing sequence of lattices which is bounded below by N_∞. It remains to prove (6). There exists a large enough integer $I \geq 0$ such that

$L_i = L_{i+1}$ for all $i \geq I$. Let $L_\infty := L_I$ be the common value. The module L_∞ is a lattice in M and, given $x \in L_\infty$, we have $x \in L_{I+1}$, and thus

$$\tau_M(\tau^\star x) \in L_I\left[\frac{1}{t-\theta}\right] = L_\infty\left[\frac{1}{t-\theta}\right].$$

Therefore L_∞ is stable and hence a model of $\underline{M}$. Hence $L_\infty \subset M_\mathcal{O}$, implying *a fortiori* that $L_\infty\big[\mathfrak{p}_1^{-1},\dots,\mathfrak{p}_{s-1}^{-1}\big] \subset M_\mathcal{O}\big[\mathfrak{p}_1^{-1},\dots,\mathfrak{p}_{s-1}^{-1}\big]$. The reverse inclusion follows from (4). ☐

We may now iterate the construction. By Lemma 2.1.2, L_∞ is a model of $\underline{M}$ which agrees with $M_\mathcal{O}$ after inverting $\mathfrak{p}_1,\dots,\mathfrak{p}_{s-1}$. We can then set $N = L_\infty$ and go back to step 2.

2.1.2. *Algorithm and proof of freeness.* We now explain how the construction presented above can be turned into an actual algorithm. As a byproduct, we shall obtain an alternative proof of the freeness of $M_\mathcal{O}$ avoiding the use of Quillen-Suslin's theorem.

For simplicity, we assume that $\underline{M} = (M, \tau_M)$ is effective; this hypothesis is armless because any model of (M, τ_M) is a model of $(M, (t-\theta)^h\tau_M)$ for any h, and *vice versa*. The maximal model of $(M, (t-\theta)^h\tau_M)$ then does not depend on h.

We proceed step by step.

Step 1. We assume that $\underline{M} = (M, \tau_M)$ is given by the matrix $\Phi \in M_r\big(\mathbb{F}(\theta)[t]\big)$ of τ_M in some basis $\mathbf{b}$. If f is a common denominator of the entries of Φ, the $\mathbb{F}[t,\theta]$-module spanned by $f\mathbf{b}$ is a model N of M. The matrix of τ_M in the corresponding basis is $f^{q-1}\Phi$. Besides, we have

$$\det\Phi = (t-\theta)^h \cdot \Delta(\theta)$$

where h is some integer and $\Delta(\theta) = \Delta_N(\theta)$ is the discriminant of N. One can easily determine the latter by computing the determinant of Φ and dividing by $t-\theta$ as much as possible.

We then factor $\Delta(\theta)$ and find the places $\mathfrak{p}_1,\dots,\mathfrak{p}_s$ we shall work with afterwards.

Step 2. We need to explain how to compute a basis of N_{i+1}, knowing a basis $\mathbf{b}_i$ of N_i. Since N_i is a model, it is stable under τ_M, from what we deduce that τ_M induces a *semilinear* map $\mathfrak{p}^{-1}N_i \to \mathfrak{p}^{-q}N_i$ and so, a second semilinear map $\mathfrak{p}^{-1}N_i/N_i \to \mathfrak{p}^{-q}N_i/N_i$. Besides, by definition of N_{i+1}, we have

$$N_{i+1}/N_i = \ker\left(\tau_M : \mathfrak{p}^{-1}N_i/N_i \longrightarrow \mathfrak{p}^{-q}N_i/N_i\right).$$

The domain $\mathfrak{p}^{-1}N_i/N_i$ is obviously a free module of rank r over $\mathbb{F}_\mathfrak{p}[t]$ with basis $\mathbf{b}_i^{(1)} = \mathfrak{p}^{-1}\mathbf{b}_i \bmod N_i$. Similarly $\mathfrak{p}^{-q}N_i/N_i$ is free module of rank r over $\mathbb{F}[t]\otimes\mathbb{F}[\theta]/\mathfrak{p}(\theta)^q$. There is moreover a computable ring isomorphism $\mathbb{F}_\mathfrak{p}[u]/u^q \cong \mathbb{F}[\theta]/\mathfrak{p}(\theta)^q$: it takes u to $\mathfrak{p}(\theta)$ and any element $a \in \mathbb{F}_\mathfrak{p}$ to b^q where b is a lifting in $\mathbb{F}[\theta]$ of $a^{1/q} \in \mathbb{F}_\mathfrak{p}$. Therefore $\mathbb{F}[t]\otimes\mathbb{F}[\theta]/\mathfrak{p}(\theta)^q$ can be computationally identified with $\mathbb{F}_\mathfrak{p}[t,u]/u^q$.

Noticing the canonical embedding $\mathbb{F}_\mathfrak{p}[t] \hookrightarrow \mathbb{F}_\mathfrak{p}[t,u]/u^q$, we conclude that $\mathfrak{p}^{-q}N_i/N_i$ naturally appears as a free module over of rank qr over $\mathbb{F}_\mathfrak{p}[t]$ with basis

$$\mathbf{b}_i^{(q)} = \bigsqcup_{v=1}^{q}\left(\mathfrak{p}^{-v}\mathbf{b}_i \bmod \mathfrak{p}^{-v+1}N_i\right).$$

For these structures, the map $\tau_M : \mathfrak{p}^{-1}N_i/N_i \to \mathfrak{p}^{-q}N_i/N_i$ is semilinear with respect to the endomorphism $\tau_\mathfrak{p} : \mathbb{F}_\mathfrak{p}[t] \to \mathbb{F}_\mathfrak{p}[t]$ taking t to itself and $a \in \mathbb{F}_\mathfrak{p}$ to a^q.

Let T be the $qr\times r$ matrix representing τ_M in the bases $\mathbf{b}_i^{(1)}$ and $\mathbf{b}_i^{(q)}$. We write the Smith decomposition of T, namely $T = USV$. Here $U \in \mathrm{SL}_{qr}(\mathbb{F}_\mathfrak{p}[t])$, $V \in \mathrm{SL}_r(\mathbb{F}_\mathfrak{p}[t])$ and $S = (s_{jk})_{1\le j\le qr, 1\le k\le r}$ is a rectangular $qr\times r$ matrix with $s_{jk} = 0$ whenever $j \neq k$. Moreover, all classical algorithms for determining Smith forms present the transformation matrices U and V as a product of transvection matrices. Hence, we can not only compute them but we can also compute liftings of them, $\hat{U}$ and $\hat{V}$, lying in $\mathrm{SL}_{qr}(\mathbb{F}[t,\theta])$ and $\mathrm{SL}_r(\mathbb{F}[t,\theta])$ respectively.

For $j \in \{1,\dots,r\}$, we set $\delta_j = 1$ if $s_{jj} = 0$ and $\delta_j = \mathfrak{p}$ otherwise. Then, we claim that the columns of the product matrix

$$\hat{W} := \hat{V} \cdot \begin{pmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_r \end{pmatrix}$$

is the change-of-basis matrix from $\mathbf{b}_i$ to a basis $\mathbf{b}_{i+1}$ of N_{i+1}. In other words, the columns of the $\hat{W}$ gather the coordinates in $\mathbf{b}_i$ of the basis $\mathbf{b}_{i+1}$ we were looking for. Indeed, the columns of $\hat{V}$ corresponding the indices j with $s_{jj} = 0$ form a basis of N_{i+1}/N_i, while the remaining columns form a basis of a complement of it in $\mathfrak{p}^{-1}N_i/N_i$.

Step 3. It is quite similar to Step 2. By Lemma 2.1.2.(3), we know that $\mathfrak{p}L_i$ is stable by τ_M. Consequently, τ_M takes L_i to $\mathfrak{p}^{-q+1}L_i$ and it gives rise to a semilinear map $L_i/\mathfrak{p}L_i \longrightarrow \mathfrak{p}^{-q+1}L_i/\mathfrak{p}L_i$. The quotient space $L_{i+1}/\mathfrak{p}L_i$ then appears as a kernel as follows:

$$L_{i+1}/\mathfrak{p}L_i = \ker\left(\tau_M : L_i/\mathfrak{p}L_i \longrightarrow \mathfrak{p}^{-q+1}L_i/\mathfrak{p}L_i\right).$$

We can now reuse the machinery of Step 2 to compute an explicit basis of L_{i+1} starting from a basis of L_i.

Remark 2.1.3. Building upon this algorithm, one may compute the discriminant of $\underline{M}$ as well and, in particular, determine the set of places of bad reduction of $\underline{M}$.

2.1.3. *About complexity.* It is not easy to give precise estimations on the complexity of the algorithm depicted above because it looks difficult to control the size of the elements in $\mathbb{F}[t,\theta]$ that show up throughout the computation. One can nevertheless give bounds on the number of iterations after the sequences $(N_i)_i$ and $(L_i)_i$ stabilize. We start with $(L_i)_i$, which is the simplest.

Lemma 2.1.4. *We have $L_r = L_\infty$ where $r = \operatorname{rank} \underline{M}$.*

Proof. We deduce from Lemma 2.1.2 that $\mathfrak{p}L_0 = N_\infty \subset L_i \subset L_0$ for all i. The lemma follows after noticing that the $\mathbb{F}_\mathfrak{p}[t]$-length of the quotient $L_0/\mathfrak{p}L_0$ is r. ☐

In order to prove a similar bound for (N_i), we need estimations on discriminants. If N is a model of $\underline{M}$, we write Δ_N for its discriminant.

Lemma 2.1.5. *Let N and N' be two models of $\underline{M}$ with $\mathfrak{p}N \subset N' \subset N$. Then*

$$\Delta_N(\theta) = \Delta_{N'}(\theta) \cdot \mathfrak{p}(\theta)^{-(q-1)\operatorname{rank}_{\mathbb{F}_\mathfrak{p}[t]}(N/N')}.$$

Proof. We apply the snake lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau^\star N' & \xrightarrow{\tau_M} & (t-\theta)^{-h}N' & \longrightarrow & \operatorname{coker}(\tau_M|N') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tau^\star N & \xrightarrow{\tau_M} & (t-\theta)^{-h}N & \longrightarrow & \operatorname{coker}(\tau_M|N) & \longrightarrow & 0 \end{array}$$

where h is a large enough integer for which τ_M is well-defined. If α denotes the map between the two cokernels, we have the following exact sequences

$$0 \longrightarrow \ker\alpha \longrightarrow \operatorname{coker}(\tau_M|N') \longrightarrow \operatorname{coker}(\tau_M|N) \longrightarrow \operatorname{coker}\alpha \longrightarrow 0$$
$$0 \longrightarrow \ker\alpha \longrightarrow \tau^\star(N/N') \longrightarrow N/N' \longrightarrow \operatorname{coker}\alpha \longrightarrow 0$$

which give the desired formula upon taking Fitting ideals. ☐

Corollary 2.1.6. *Let N be the model of $\underline{M}$ constructed at Step 1 of the algorithm and write*

$$\Delta_N(\theta) = \mathfrak{p}_1(\theta)^{n_1} \cdots \mathfrak{p}_s(\theta)^{n_s}$$

where $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ are distinct places. Then, when treating the prime $\mathfrak{p}_j$, we have $N_\infty = N_{\lfloor \frac{n_j}{q-1} \rfloor}$.

Proof. The algorithm starts with the place $\mathfrak{p}_s$. By Lemma 2.1.5, the $\mathfrak{p}_s$-adic valuation of Δ_{N_i} decreases at least by $q-1$ at each iteration. The number of iterations is then bounded above by $\lfloor \frac{n_s}{q-1} \rfloor$. Then, when passing to the next place, the initial model N will be replaced by the model L_∞ we have ended with. In order to repeat the argument, we need to prove that the Δ_{L_∞} and Δ_N only differ by a power of $\mathfrak{p}_s$. This follows again from Lemma 2.1.5, given that $\mathfrak{p} L_\infty \subset N_\infty \subset L_\infty$. □

2.1.4. *The local case.* Having a global maximal model defined over $\mathbb{F}[t,\theta]$ is very nice. However, for many applications, it is often enough to know a *local* maximal model, involving only one place $\mathfrak{p}$. Formally, given a finite place $\mathfrak{p}$, we introduce the ring

$$\mathbb{F}[t,\theta]^\wedge_{\mathfrak{p}(\theta)} \coloneqq \varprojlim_n \left(\mathbb{F}[t] \otimes \mathbb{F}[\theta]/\mathfrak{p}(\theta)^n\right)$$

and extend the morphism τ to it. A t-motive $\underline{M} = (M, \tau_M)$ defines by scalar extension to $\mathbb{F}[t,\theta]^\wedge_{\mathfrak{p}(\theta)}$ a pair $(M_\mathfrak{p}, \tau_{M_\mathfrak{p}})$. There is a notion of maximal model in this new setting, and one checks that the maximal model of $(M_\mathfrak{p}, \tau_{M_\mathfrak{p}})$ is simply $M_{\mathscr{O},\mathfrak{p}} \coloneqq \mathbb{F}[t,\theta]^\wedge_{\mathfrak{p}(\theta)} \otimes_{\mathbb{F}[t,\theta]} M_\mathscr{O}$ (*e.g.* [Gaz, Theorem 4.55]).

Of course, the knowledge of $M_{\mathscr{O},\mathfrak{p}}$ is much weaker than that of $M_\mathscr{O}$. However, it is enough to decide if $\underline{M}$ has good reduction at $\mathfrak{p}$ and, more generally, to determine the local factor at $\mathfrak{p}$. Indeed we have the following theorem which is an immediate consequence of Theorem 1.2.5.

Theorem 2.1.7. *For any two distinct finite places $\mathfrak{p}$ and ℓ of A, we have $P_\mathfrak{p}(\mathrm{T}_\ell \underline{M}, T) = P_\mathfrak{p}(M_{\mathscr{O},\mathfrak{p}}, T)$.*

On the other hand, we claim that the algorithm presented in §2.1.2 works similarly with $(M_\mathfrak{p}, \tau_{M_\mathfrak{p}})$ with two important simplifications:

- there is a unique place which matters, namely $\mathfrak{p}$,
- the matrix $\hat{V}$ can be any lift of V in $\mathrm{Mat}_r(\mathbb{F}[t,\theta]^\wedge_\mathfrak{p})$; any such lift will be automatically invertible and will do the job.

As a consequence of the last point, we no longer need to compute $\hat{V}$ at the same time of V and, more importantly, we can choose its entries as polynomials in t and θ of controlled degrees. This results in a faster algorithm for which one can hope nice bounds on the complexity.

2.2. Computation of the L-series.

Let v be a place of K (finite or infinite). In this subsection, we explain how to compute the L-series of $\underline{M}$ from Anderson trace formula.

As we observed in Subsection 1.3, Anderson trace formula does not apply readily to $\underline{M} = (M, \tau_M)$. This is because τ_M does not generally define an $\mathbb{F}[t]$-linear endomorphism of M. Instead, the strategy consists in applying the trace formula modulo a power of v.

For better clarity, we write $\tau_\theta = \tau$ and we equip $\mathbb{F}[t,\theta]$ with another endomorphism τ_t given as the $\mathbb{F}[\theta]$-linear morphism $\tau_t(t) = t^q$. We observe that τ_t and τ_θ commute and that their composition is the q-Frobenius on $\mathbb{F}[t,\theta]$.

We let $\mathbb{F}(t,\theta)$ be the fraction field of $\mathbb{F}[t,\theta]$; the endomorphisms τ_t and τ_θ uniquely extend to $\mathbb{F}(t,\theta)$.

2.2.1. *The map $\tau_M^\star$.* Under the identification $\Omega^1_{\mathbb{F}[\theta]} = \mathbb{F}[\theta]d\theta$, we express the action of the $\mathbb{F}[t]$-linear extension of the q-Cartier operator $C = C_\theta : \mathbb{F}[t,\theta] \to \mathbb{F}[t,\theta]$ as follow. It is given by the formula:

$$C_\theta\left(\sum_{i,j\geq 0} a_{i,j}t^i\theta^j\right) = \sum_{i,j\geq 0} a_{i,qj+q-1}t^i\theta^j.$$

In other words, we solely select in the polynomial expansion coefficients whose degree in θ has residue $q-1$ modulo q. Note that it has the following action on the degree in θ:

$$\text{if} \quad \deg_\theta p(t,\theta) \leq d, \quad \text{then} \quad \deg_\theta C_\theta(p(t,\theta)) \leq \frac{d}{q} - \frac{q-1}{q}. \tag{8}$$

Implicitly, we made the identification $\sigma^\star\Omega^1_{\mathbb{F}[\theta]} = \Omega^1_{\mathbb{F}[\theta]}$ at the cost of rendering C_θ only τ_θ^{-1}-semilinear; *i.e.* it verifies the relation $C_\theta(\tau_\theta(x)y) = xC_\theta(y)$ for all $x, y \in \mathbb{F}[t,\theta]$. This observation allows to extend the Cartier operator to the fraction field of $\mathbb{F}[t,\theta]$ by mean of the formula

$$C_\theta\left(\frac{x}{y}\right) = \frac{C_\theta(xy^{q-1})}{\tau_t(y)}.$$

Let $M^\star$ denote the algebraic dual of M, *i.e.* $M^\star = \mathrm{Hom}_{\mathbb{F}[t,\theta]}(M, \mathbb{F}[t,\theta])$. As in Subsection 1.3, the map τ_M defines a *dual map* $\tau_M^\star$ via the formula

$$\tau_M^\star : M^\star \longrightarrow M^\star \otimes_{\mathbb{F}[t,\theta]} \mathbb{F}(t,\theta), \quad f \longmapsto C_\theta \circ f \circ \tau_M$$

(it is only $\mathbb{F}[t]$-linear). One should be aware that τ_M will not preserve $M^\star$ nor $M^\star[(t-\theta)^{-1}]$ in general, explaining why Anderson trace formula does not apply directly to (M, τ_M).

We may circle this issue depending on whether the place v is finite or infinite. Let n be a positive integer.

(1) If the place v is finite, τ_M defines an A/v^n-linear endomorphism of

$$M_{\mathscr{O}} \otimes_{A\otimes A} \left(A/v^n \otimes A[v^{-1}]\right) \tag{9}$$

(as $(t-\theta)$ becomes invertible in $A/v^n \otimes A[v^{-1}]$). Setting $B = A/v^n$ and $R = A[v^{-1}]$, we may apply Anderson trace formula to the free $B \otimes R$-module (9).

(2) If, otherwise, $v = \infty$ is the infinite place, then we can do as follow. We introduce the rings

$$\mathscr{A}_\infty(A) \coloneqq \varprojlim_i (\mathscr{O}_\infty/\mathfrak{m}_\infty^i \otimes A) = \mathbb{F}[\theta][[1/t]], \quad \mathscr{B}_\infty(A) \coloneqq K_\infty \otimes_{\mathscr{O}_\infty} \mathscr{A}_\infty(A) = \mathbb{F}[\theta]((1/t)).$$

Next, we fix a finite free $\mathscr{A}_\infty(A)$-submodule Λ_∞ of $\mathscr{B}_\infty(M_{\mathscr{O}}) \coloneqq M_{\mathscr{O}} \otimes_{A\otimes A} \mathscr{B}_\infty(A)$ (*e.g.* the $\mathscr{A}_\infty(A)$-module generated by a basis of $M_{\mathscr{O}}$ over $\mathbb{F}[t,\theta]$). There exists k large enough for which $t^{-k}\tau_M(\tau^\star\Lambda_\infty) \subset \Lambda_\infty$. Setting $B = \mathscr{O}_\infty/\mathfrak{m}_\infty^n$ and $R = A$, we may now apply Anderson trace formula to the pair $(\Lambda_\infty/\mathfrak{m}_\infty^n\Lambda_\infty, t^{-k}\tau_M)$.

2.2.2. *Nucleus for $\tau_M^\star$.* We should now find a nucleus respectively to the above situations. Let $h \in \mathbb{Z}$ be an integer for which

$$\tau_M(\tau^\star M) \subset \frac{1}{(t-\theta)^h} M$$

(typically the smallest one). We let $h_0 \coloneqq h$ and define $(h_i)_{i\geq 0}$ by induction using the formula $h_{i+1} = \lceil \frac{h_i}{q} \rceil$.

We start with the case of a finite place v. Let $a \in \mathbb{F}_v \otimes 1$ be the image of t under the quotient map $A \to \mathbb{F}_v$. We interpret a in $A_v \otimes A$, so that $v(\theta) = (\theta - a)(\theta - a^q)\cdots(\theta - a^{q^{d-1}})$ where $d = \deg v$.
Let $c \geq 0$ be an integer, which we assume to be divisible by d for simplicity.

Lemma 2.2.1. *There exist d integers $k_0, \dots, k_{d-1}$ such that*

$$w_0 := qk_1 - k_0 - h_c,\ w_1 := qk_2 - k_1,\ w_2 := qk_3 - k_2,\ \dots,\ w_{d-2} := qk_{d-1} - k_{d-2},\ w_{d-1} := qk_0 - k_{d-1}$$

are all in the range $[0, q-1]$.

Proof. We set $k_0 = \left\lceil \frac{h_c}{q^d-1} \right\rceil$ and $k_i = \left\lceil \frac{q^{d-i}h_c}{q^d-1} \right\rceil$ for $1 \le i \le d-1$. A direct computation shows that

$$w_i = q\left\lceil \frac{q^{d-i-1}h_c}{q^d-1} \right\rceil - \left\lceil \frac{q^{d-i}h_c}{q^d-1} \right\rceil$$

for all i. The lemma follows after noticing that $q\lceil x \rceil - \lceil qx \rceil$ is in $[0, q-1]$ for all real number x. □

We form the products

$$\delta := \frac{1}{v(\theta)} \cdot \prod_{i=0}^{d-1} \left(a^{q^i} - \theta\right)^{-k_i} \cdot \prod_{i=1}^{c} \left(t^{q^i} - \theta\right)^{-h_i},$$

$$\rho := v(\theta)^{q-1} \cdot \prod_{i=0}^{d-1} \left(a^{q^i} - \theta\right)^{w_i} \cdot \prod_{i=0}^{c-1} \left(t^{q^i} - \theta\right)^{qh_{i+1}-h_i}.$$

By design, δ and ρ verifies the following relation:

$$\frac{1}{(t-\theta)^h} \cdot \left(\frac{t^{q^c} - \theta}{a - \theta}\right)^{h_c} \delta = \rho \cdot \tau_\theta(\delta). \tag{10}$$

Fix a basis $(e_1, \dots, e_r)$ of $M_{\mathcal{O}}$ over $\mathbb{F}[t, \theta]$ and denote by $(e_1^\star, \dots, e_r^\star)$ the dual basis of $M^\star$. We denote by $(b_{ij})_{1 \le i, j \le r} \in \mathrm{Mat}_r(\mathbb{F}[t, \theta])$ the matrix representing $(t-\theta)^h \tau_M$ in $(e_1, \dots, e_r)$. For indices $i, j \in \{1, \dots, r\}$ and $x \in \mathbb{F}(t, \theta)$, we have

$$\tau_M^\star(x e_i^\star)(e_j) = C_\theta\left(\frac{x b_{ij}}{(t-\theta)^h}\right). \tag{11}$$

Let $R = \mathbb{F}[\theta][v(\theta)^{-1}]$ as above.

Lemma 2.2.2. *For all $x \in \mathbb{F}(t, \theta)$, we have*

$$\tau_M^\star\left(x \delta e_i^\star\right) \equiv \sum_{j=1}^{r} C_\theta\left(\rho \cdot x \cdot b_{ij}\right) \delta e_j^\star \pmod{v(t)^{q^c} \otimes R}$$

Proof. Since d divides c, the fraction $\frac{t^{q^c} - \theta}{a - \theta}$ is in $1 + v(t)^{q^c} \otimes R$. The lemma follows easily combining Equations (10) and (11). □

To apply Anderson formula, we put ourselves in the situation of Theorem 1.3.4 with $B = \mathbb{F}[t]/v(t)^{q^c}$. We consider the finite free $B \otimes R$-module

$$N := M_{\mathcal{O}} \otimes_{A \otimes A} (B \otimes R),$$

a basis of which being $(e_1, \dots, e_r)$.

Lemma 2.2.3. *The element δ is a unit in $B \otimes R$.*

Proof. We observe that

$$\delta \equiv \frac{1}{v(\theta)} \cdot \prod_{i=0}^{d-1} \left(a^{q^i} - \theta\right)^{-k_i} \cdot \prod_{i=1}^{c} \left(a^{q^i} - \theta\right)^{-h_i} \pmod{v(t) \otimes R}.$$

Hence δ is a divisor of a power of $v(\theta)$ in $\mathbb{F}[t]/v(t) \otimes R \simeq \mathbb{F}_v[\theta][1/v(\theta)]$. It is then a unit in this ring. By Hensel lemma, it is also a unit in $B \otimes R$. ☐

It follows from the lemma that $(\delta e_1, \ldots, \delta e_r)$ is also a basis of N. We let $N_0 \subset N$ be the free R-module generated by $(\delta e_1, \ldots, \delta e_r)$ and let $N_0^\star \subset N^\star$ be the R-module generated by $(\delta e_1^\star, \ldots, \delta e_r^\star)$, so that we have the identifications

$$N = B \otimes N_0, \quad \text{and} \quad N^\star = \mathrm{Hom}_{B\otimes R}(N, B \otimes \Omega^1_R) = B \otimes N_0^\star.$$

Lemma 2.2.4. *Let* $s_{\max} \coloneqq \lceil \frac{d_\theta}{q-1} \rceil + 2d + c$ *where* $d_\theta = \max_{i,j} \deg_\theta(b_{ij})$. *Then, the* $\mathbb{F}$*-subvector space of* $N_0^\star$ *given by*

$$W_0 \coloneqq \left\langle \left\{ \theta^s \cdot \delta e_j^\star \;\middle|\; s \in \{0, \ldots, s_{\max}\},\; j \in \{1, \ldots, r\} \right\} \right\rangle_{\mathbb{F}} \tag{12}$$

is a nucleus for $\tau_M^\star : N^\star \to N^\star$.

Proof. Given two nonnegative integers a and b, we set

$$N^\star_{a,b} \coloneqq \left\langle \left\{ v(\theta)^{-\alpha}\theta^\beta \cdot \delta e_j^\star \;\middle|\; \alpha \in \{0, \ldots, a\},\; \beta \in \{0, \ldots, b\},\; j \in \{1, \ldots, r\} \right\} \right\rangle_B.$$

The union of the $N^\star_{a,b}$ is clearly the entire space $N^\star$. Applying Lemma 2.2.2 with $x = v(\theta)^{-\alpha}\theta^\beta$, we get the congruence

$$\begin{aligned}\tau_M^\star\left(v(\theta)^{-\alpha}\theta^\beta \delta e_i^\star\right) &\equiv \sum_{j=1}^r C_\theta\left(\rho \cdot v(\theta)^{-\alpha}\theta^\beta \cdot b_{ij}\right)\delta e_j^\star \\ &\equiv v(\theta)^{-\alpha'} \sum_{j=1}^r C_\theta\left(\rho \cdot v(\theta)^{q\alpha'-\alpha}\theta^\beta \cdot b_{ij}\right)\delta e_j^\star \pmod{v(t)^{q^c} \otimes R}\end{aligned}$$

We choose $\alpha' = \left\lfloor \frac{\alpha}{q} \right\rfloor$. The exponent $q\alpha' - \alpha$ is then in the range $[1-q, 0]$, showing that the expression inside C_θ is a polynomial in $\mathbb{F}[t, \theta]$ whose θ-degree does not exceed $(q-1)(2d+c) + \beta + d_\theta$. Applying C_θ then gives by (8) a polynomial whose θ-degree is at most $\frac{\beta + d_\theta}{q} + \frac{q-1}{q}(2d+c)$. Hence, we get the inclusion

$$\tau_M^\star\left(N^\star_{a,b}\right) \subset N^\star_{a',b'} \quad \text{with} \quad a' = \left\lfloor \frac{a}{q} \right\rfloor, \quad b' = \left\lfloor \frac{b + d_\theta}{q} + \frac{q-1}{q}(2d+c) \right\rfloor$$

This shows that, starting from $x \in N^\star_{a,b}$, applying $\tau_M^\star$ enough times will eventually yields a result in $N^\star_{0,s_{\max}} = B \otimes W_0$. ☐

For the place $v = \infty$, the method is very similar. We take for Λ_∞ the free $\mathbb{F}[\theta][[1/t]]$-module generated by the basis $(e_1, \ldots, e_r)$ so that $t^{-(d_t - h)}\tau_M$ stabilizes Λ_∞, where $d_t = \max_{i,j} \deg_t(b_{ij})$. This time, we rather consider the infinite products

$$\delta \coloneqq \prod_{i=1}^\infty \left(1 - \frac{\theta}{t^{q^i}}\right)^{-h_i}, \quad \rho \coloneqq \prod_{i=0}^\infty \left(1 - \frac{\theta}{t^{q^i}}\right)^{qh_{i+1} - h_i},$$

which now satisfy the relation

$$\frac{\delta}{\left(1 - \frac{\theta}{t}\right)^h} = \rho\tau_\theta(\delta).$$

The relevant nucleus for $t^{-(d_t-h)}\tau_M^\star$ acting on $\mathrm{Hom}_{\mathbb{F}[\theta][[1/t]]}(\Lambda_\infty, \mathbb{F}[\theta][[1/t]]/(1/t)^{q^c})$ is now the finite dimensional $\mathbb{F}$-vector space

$$W_0 \coloneqq \left\langle \left\{ \theta^s \cdot \delta e_j^\star \;\middle|\; s \in \{0, \ldots, s_{\max}\},\; j \in \{1, \ldots, r\} \right\} \right\rangle_{\mathbb{F}}. \tag{13}$$

Input:

- the matrix $\Phi \in \mathrm{Mat}_r\big(\mathbb{F}[t,\theta]\big)$ giving the action of $(t-\theta)^h\tau_M$ on a maximal model $M_{\mathscr{O}}$ of $\underline{M}$
- a place v of $\mathbb{F}[t]$, represented either by the symbol ∞ or by an irreducible polynomial
- a target precision prec

Output:

- the L-series $L_v(\underline{M},T)$ at precision v^{prec}

Case of a finite place

1. Compute the smallest integer c such that d divides c and $q^c \geq \mathrm{prec}$

2. Form the nucleus W_0 defined by Equation (12)

3. Compute the matrix $\Phi^\star \in \mathrm{Mat}_{\dim_{\mathbb{F}} W_0}(\mathscr{O}_v)$ giving the action of $\tau_M^\star$ on $\mathscr{O}_v \otimes W_0$ using the explicit formula of Lemma 2.2.2

4. Return $\det(1 - T\Phi^\star) + O(v^{\mathrm{prec}})$

Case $v = \infty$

1. Compute the smallest integer c such that $q^c - r\cdot(d_t - h)\cdot\big(c + \frac{d_\theta}{q-1}\big) \geq \mathrm{prec}$

where d_t (resp. d_θ) is the maximal t-degree (resp. θ-degree) of an entry of Φ

2. Form the nucleus W_0 defined by Equation (13)

3. Compute the matrix $\Phi^\star \in \mathrm{Mat}_{\dim_{\mathbb{F}} W_0}(\mathscr{O}_v)$ giving the action of $t^{-(d_t-h)}\tau_M^\star$ on $\mathscr{O}_v \otimes W_0$ using the explicit formula of Lemma 2.2.2

4. Compute $\chi(U) = \det(1 - U\Phi^\star) + O\big(t^{-q^c}\big)$ and return $\chi(t^{d_t-h}T) + O(t^{-\mathrm{prec}})$

FIGURE 1. Algorithm for computing L-series of a t-motive

with $s_{\max} = c + \left\lfloor \frac{d_\theta}{q-1} \right\rfloor$

2.2.3. *The algorithm.* The above discussion translates readily to an algorithm computing the L-series of an Anderson t-motive $\underline{M}$ at some arbitrary precision, provided that we know a maximal model. The algorithm is outlined in Figure 1.

Its correctness follows from Anderson trace formula together with what we have done in §2.2.2.

Before discussing the complexity, we need to clarify how we represent the ring $\mathscr{O}_v$ and how we perform computations in it. By definition $\mathscr{O}_v$ is the completion of $\mathbb{F}[\theta]$ at the place v. Since $\mathscr{O}_v$ has equal characteristic, it can be identified with a ring of Laurent series over the residue field $\mathbb{F}_v$. Concretely an isomorphism realizing this identification is, for example:

$$\iota_v : \mathscr{O}_v \xrightarrow{\sim} \mathbb{F}_v[[u]], \quad \theta \mapsto a + u$$

where $a \in \mathbb{F}_v$ is a root of $v(\theta)$. Computing $\iota_v(f(\theta))$ for $f(\theta) \in \mathbb{F}[\theta]$ then amounts to shifting the polynomial f, which can be achieved in quasilinear time in the degree of f. Similarly, all field operations (addition, subtraction, multiplication, division) in $\mathbb{F}_v[[u]]$ at precision u^{prec} can be performed in quasilinear time in the precision.

We recall that computing a characteristic polynomial of a $d\times d$ matrix over $\mathbb{F}[\theta]$ with entries known at precision prec can be achieved for a cost of $O^{\sim}(d^{\Omega}\cdot \mathrm{prec})$ operations in $\mathbb{F}$ with $\Omega < 2.69497$ using [KaV]. In what precedes, we used the standard notation $O^{\sim}$ meaning that logarithmic factors are hidden.

Theorem 2.2.5. *Algorithm of Figure 1 requires at most*

$$O^{\sim}\left(r^{\Omega}\cdot\left(\frac{d_\theta}{q-1}+d\right)^{\Omega}\cdot d\cdot \mathrm{prec}\right)$$

operations in $\mathbb{F}$.

Proof. Computing the matrix $\Phi^\star$ in step 3 amounts to computing the products $\rho{\cdot}b_{ij}$ in $\mathbb{F}_v[[u]]$ (for i,j varying between 1 and r) and selecting the relevant coefficients. This can be achieved for a cost of $O^{\sim}(r^2 d\cdot\mathrm{prec})$ operations in $\mathbb{F}$.
Coming back to the definition, we see that $\dim_{\mathbb{F}} W_0 \in O\big(r\cdot\big(\frac{d_\theta}{q-1}+d+c\big)\big)$. Hence the computation of the characteristic polynomial in step 4 requires no more than

$$O^{\sim}\left(r^{\Omega}\cdot\left(\frac{d_\theta}{q-1}+d+c\right)^{\Omega} d\cdot \mathrm{prec}\right)$$

operations in $\mathbb{F}$. The announced complexity follows after observing than $c = O(\log \mathrm{prec})$. □

The rather surprising fact that we can achieve quasilinear complexity with respect to the precision is a consequence of the extremely rapid v-adic convergence of the L-series $L_v(\underline{M},T)$: in order to get an accurate result at precision v^{prec}, one only needs $O(\log\mathrm{prec})$ terms in the expansion of the series. This amazing property can be rephrased in analytic terms as follows.

Theorem 2.2.6. *Let v be a finite or infinite place of $\mathbb{F}[t]$. Write $L_v(\underline{M},T) = \sum_i a_n T^n$ for coefficients $a_n \in A_v$. Then,*

$$|a_n|_v = O\left(q^{-\deg v\cdot c^n}\right)$$

where $c = q^{1/(\mathrm{rank}\,\underline{M}\cdot\deg v)} > 1$ and $|\cdot|_v = q^{-\deg v\cdot v_v(\cdot)}$. In particular, the radius of convergence of $L_v(\underline{M},T)$ is infinite.

Another remarkable fact is that the parameter h does not appear in the complexity. This is again the algorithmical counterpart of an interesting theoretical result of continuity. Before stating it, we define the Carlitz twist of a t-motive: if $\underline{M} = (M,\tau_M)$ is a t-motive and h is an integer, we set $\underline{M}(h) = (M,(t-\theta)^h\tau_M)$.

Theorem 2.2.7. *Let v be a finite place of $\mathbb{F}[t]$ of degree d. Let h,h' be two integers such that $h\equiv h'$ $(\mathrm{mod}\ (q^d-1)q^c)$ for some nonnegative integer c. Then*

$$L_v\big(\underline{M}(h),T\big) \equiv L_v\big(\underline{M}(h'),T\big) \pmod{v^{q^c}}.$$

Proof. This follows from the observation that our algorithm computes the L-series $L_v\big(\underline{M}(h),T\big)$ modulo v^{q^c}, by extracting from h only the following values:

$$qh_1-h_0,\ qh_2-h_1,\ \ldots,\ qh_c-h_{c-1},\ \left\lceil\frac{h_c}{q^d-1}\right\rceil,\ \left\lceil\frac{qh_c}{q^d-1}\right\rceil,\ \ldots,\ \left\lceil\frac{q^{d-1}h_c}{q^d-1}\right\rceil.$$

One checks that all the preceding values depend only on the class of h modulo $(q^d-1)q^c$. □

2.2.4. *A remark on the maximal model.* The algorithm of Figure 1 assumes that the maximal model $M_{\mathcal{O}}$ of $\underline{M}$ is known. If it is not the case, one can compute it using the algorithm of Subsection 2.1. However, this method could be quite slow as the description of the maximal model we obtain this way could involve polynomials in t and θ of very large degrees.

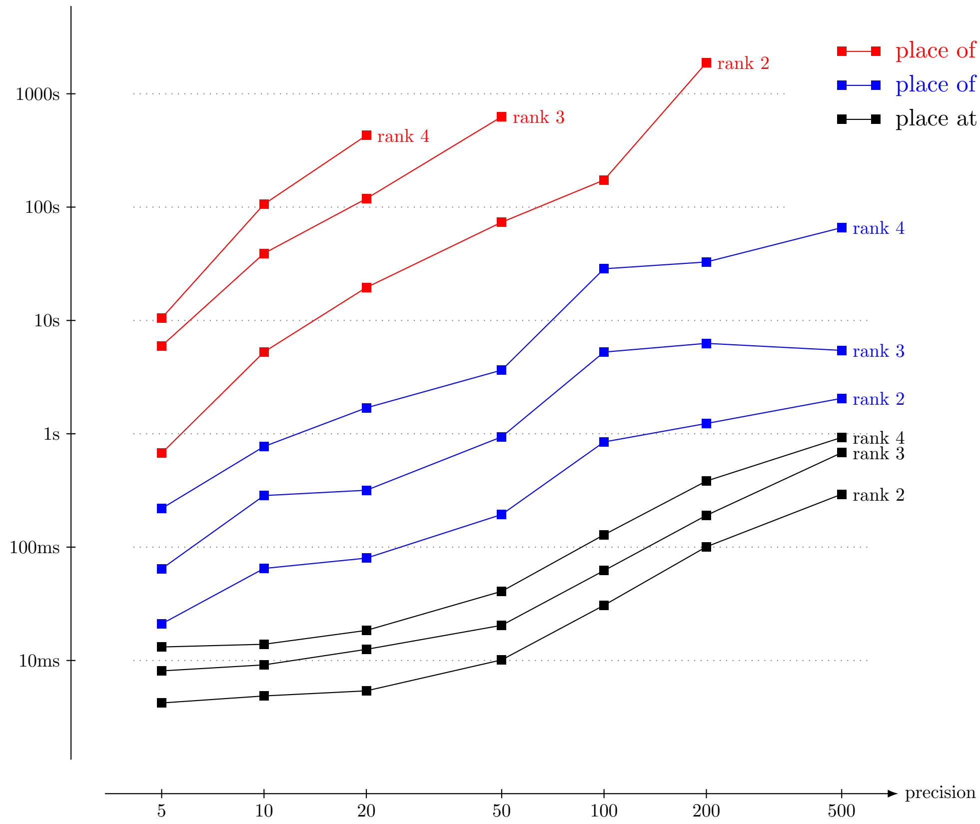

FIGURE 2. Timings for the computation of the L-series with $q = d_\theta = 9$.
(CPU: Intel Core i5-8250U at 1.60GHz — OS: Ubuntu 22.04.1)

A better option might be the following. We start with a nonmaximal model M. We compute its discriminant $\Delta_M \in \mathbb{F}[\theta]$ and factor it: $\Delta_M = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_s^{n_s}$. Then, on the one hand, we set $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ and we use (a variant of) the algorithm of Figure 1 to compute the L-series outside S, namely

$$L_{S,v}(\underline{M}, T) = \prod_{\substack{\mathfrak{p} \notin S \\ \mathfrak{p} \neq v}} P_{\mathfrak{p}}(M, T)^{-1}.$$

On the other hand, we compute the *local* maximal models $M_{\mathcal{O},\mathfrak{p}_1}, \ldots, M_{\mathcal{O},\mathfrak{p}_s}$ using the method sketched in §2.1.4 and, for each of them, we compute the local factor $P_{\mathfrak{p}_i}(M_{\mathcal{O},\mathfrak{p}_i}, T)$. Finally, we recombine the results to get the desired L-series:

$$L_v(\underline{M}, T) = L_{S,v}(\underline{M}, T) \cdot \prod_{i=1}^{s} P_{\mathfrak{p}_i}(M_{\mathcal{O},\mathfrak{p}_i}, T)^{-1}.$$

3. IMPLEMENTATION AND TIMINGS

We have implemented in SageMath [SAGE] the algorithm of Figure 1, *i.e.* the computation of the L-series assuming that we can given the maximal model. Our package is publicly available on gitlab at the address

https://plmlab.math.cnrs.fr/caruso/anderson-motives

Figure 2 gives an overview on the timings obtained with our package when the precision, the rank and the degree of the place vary. For fast computations (less than 10 seconds), the timings displayed in the figure are averaged on many runnings (up to 100) of the algorithm with various inputs. They are therefore quite reliable. On the contrary, slow computations were run only once; hence the reader should consider the corresponding timings with extreme caution.

We observe that the dependence in the precision is rather good, almost quasilinear as predicted by Theorem 2.2.5. It is also striking that the degree of the underlying place has a strong impact on the timings, much higher than the rank of the Anderson motive. Again, this is in line with the theoretical complexity given in Theorem 2.2.5 although the effect seems even more pronounced on the timings. This could be due to the fact that computations in the completions A_v stays slow.

We notice also that the computation for the place at infinity is much faster (by a factor 10 about) than for any other place of degree 1. This can be explained by the fact that, when working at the place at infinity, the size of the nucleus can be lowered a bit according to the t-degrees of the entry of Φ, an optimization we have implemented in our package.

Finally, we underline that, in all cases, most of the time is spent in the final computation of the characteristic polynomial, for which we rely on the implementation included in SageMath. Unfortunately, the latter is not optimized (SageMath uses a quartic algorithm) and is rather slow. Improving this basic routine would then mecanically result in a significant speed up of the computation of the L-series.

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